MAT 322/523 Lec 13 W Mar 6

<u>Recall</u>: • $g: A \to \mathbb{R}^n$ <u>diffeo.</u> ($A \overset{\text{open}}{\subset} \mathbb{R}^n$)

iff g injective & $\det Dg \neq 0$ on A.

• $A, B \overset{\text{open}}{\subset} \mathbb{R}^n$, $g: A \to B$, $f: B \to \mathbb{R}^n$ continuous

$$\int_B f = \int_A (f \circ g) \, |\det Dg|$$

We now prove the change of variable theorem in $n=1$:

<u>Thm</u>: $I = [a,b]$, $g: I \to \mathbb{R}$ C^∞

: $g'(x) \neq 0$ on (a,b), $f: g(I) \to \mathbb{R}$

continuous, then

$$\int_J f = \int_I (f \circ g) g'.$$

<u>Proof</u>: g is C^∞ so g' is continuous.

Intermediate value thm implies

$g'(x) > 0$ or $g'(x) < 0$ $\forall x \in (a,b)$

$\Rightarrow g$ strictly monotonic on I

$\Rightarrow g$ injective

$J = [g(a), g(b)]$ $\left(\text{or } [g(b), g(a)]\right)$

$\boxed{\text{if } g'>0}$ wlog assume this $\qquad$ if $g'<0$

IVT again implies $g: I \to J$ surjective.

so $f(g(x))$ def for all $x \in [a,b]$.

Def $F(y) = \int_{g(a)}^{y} f$. Fundamental thm of calculus (f continuous) $\Rightarrow F'(y) = f(y)$

$h(x) = F(g(x))$. Chain rule

$$h'(x) = F'(g(x)) \cdot g'(x) = f(g(x))\, g'(x)$$

$$\int_a^b h'(x) = \int_a^b f(g(x))\, g'(x) = h(b) - h(a)$$

$$= F(g(b)) - F(g(a)) = \int_c^{g(b)} f - \int_c^{g(a)} f = \int_{g(a)}^{g(b)} f \qquad \square$$

§18 in Munkres

Def: $A, B \overset{\text{open}}{\subset} \mathbb{R}^n$ $(n \geq 2)$ and $h: A \to B$ diffeo is called a primitive diffeo if it is of the form

$$h(\vec{x}) = (h_1(\vec{x}), \dots, h_n(\vec{x}))$$ such that $h_i(\vec{x}) = x_i$ for some i

---

Thm: $A, B \overset{\text{open}}{\subset} \mathbb{R}^n$ $(n \geq 2)$, $g: A \to B$ diffeo.
Given $\vec{a} \in A$ $\exists$ nghd $U_0 \subset A$ of $\vec{a}$ and a sequence of primitive diffeos

$$U_0 \xrightarrow{h_1} U_1 \xrightarrow{h_2} \cdots \xrightarrow{h_k} U_k$$

such that $h_k \circ \cdots \circ h_1 = g|_{U_0}$.

Rmk: This theorem says that a diffeo locally looks like the composition of primitive diffeos

— this is the main reason we introduced partitions of unity.

Recall 1/2 of the change of variables thm:

<u>Lma</u> $A, B \overset{\text{open}}{\subset} \mathbb{R}^n$, $g: A \to B$, $f: B \to \mathbb{R}^n$ continuous

Then:

f integrable on B $\Rightarrow$ $(f \circ g)|\det Dg|$ integrable on A.

$$\& \int_B f = \int_A (f \circ g)|\det Dg|$$

<u>Proof sketch</u>: For all $a \in A$ pick $U_a \subset A$ so that

$g|_{U_a}$ = composition of primitive diffeos.

Then $\{U_a\}_{a \in A}$ and $\{g(U_a)\}_{a \in A}$ are open covers of A and B, resp.

Choose a partition of unity $\{\phi_a\}_{a\in A}$ of $\{g(U_a)\}_{a\in A}$ w/ compact support.

Then $\{\phi_a \circ g\}_{a\in A}$ is a partition of unity of $\{U_a\}_{a\in A}$ w/ compact support.

Then

$$\int_B f = \sum_{a\in A}\left(\int_{g(U_a)} \phi_a f\right)$$

Want to prove that this equal to

$$\sum_{a\in A}\left(\int_{U_a}[(\phi_a f)\circ g|_{U_a}]|\det g|_{U_a}|\right)$$

Now we can decompose

$g|_{U_a} = h_k^a \circ \dots \circ h_1^a$ into primitive diffeos.

<u>Claim</u>: If $U_0 \xrightarrow{h_1} U_1 \xrightarrow{h_2} U_2$ are two primitive diffeos

and $f: U_2 \to \mathbb{R}$ integrable

such that $\int_{h_1(U_0)} f = \int_{U_0} (f\circ h_1)|\det Dh_1|$

$$\int_{h_2(U_1)} f = \int_{U_1} (f \circ h_2) |\det Dh_2| \quad \underline{\text{then}}$$

$$\int_{(h_2 \circ h_1)(U_0)} f = \int_{U_0} (f \circ h_2 \circ h_1) |\det D(h_2 \circ h_1)|$$

Assuming the claim is true, & looking at (*) it suffices to prove

$$\boxed{\int_{g(U_a)} f = \int_{U_a} (f \circ h_i^a) |\det Dh_i^a|} \text{ for some } i \quad (**)$$

Now, we already proved the thm for $n=1$.

<u>Induction on n</u>:

Suppose (**) holds in dim $n-1$. Then we wish to prove it for primitive diffeos of dim n.

$U, V \overset{\text{open}}{\subset} \mathbb{R}^n$, $g: U \to V$ primitive diffeo

$f: V \to \mathbb{R}$ continuous.

Let $p \in U$, $q := g(p) \in V$. Choose rectangle $Q \subset B$ containing q. wlog assume f has cpt supp in int Q (else it's another partition of unity argument). Let $S := g^{-1}(Q)$

So $F := (f \circ g)|\det Dg|$ will have compact support in S.

Want to show $\int_Q f = \int_S F$

Note $Q = Q' \times [a_n, b_n]$

rectangle in $\mathbb{R}^{n-1}$

Assume that $g: U \to V$ is so that

$$g(\vec{x}) = (g_1(\vec{x}), \dots, g_{n-1}(\vec{x}), x_n).$$

then $S \subset S' \times [a_n, b_n]$ for

Some $S' \subset \mathbb{R}^{n-1}$ <u>rectangle</u>.

Extend f, F to be equal to 0 outside of Q, S if needed.

Want to show

$$\int_Q f = \int_{S' \times [a_n, b_n]} F \overset{\text{Fubini}}{\Longleftarrow}$$

$$\int_{a_n}^{b_n} \int_{Q'} f(\vec{y}, t) = \int_{a_n}^{b_n} \int_{S'} F(\vec{y}, t)$$

$\vec{y} = (x_1, \dots, x_{n-1})$.

But to show this, it suffices to show

$$\int_{Q'} f(\vec{y}, t) = \int_{S'} F(\vec{y}, t)$$

$Q', S' \subset \mathbb{R}^{n-1}$. We skip details, but the induction hypothesis can be used here.

$\square$

<u>Lma</u>: $g: A \to B$ diffeo, $A, B \overset{\text{open}}{\subset} \mathbb{R}^n$
$f: B \to \mathbb{R}$. If $(f \circ g)|\det Dg|$ integrable on A, then f integrable on B.

<u>Proof</u>: Apply previous lemma to g^{-1} (which we know exists & is C^∞)

$F = (f \circ g)|\det Dg|$ continuous on A, integrable on A by assumpt.

$\Rightarrow$ $(F \circ g^{-1})|\det Dg^{-1}|$ integrable over B. But if $\vec{y} = g(\vec{x})$ we have

$(F \circ g^{-1}(\vec{y})) = F \circ (g^{-1}(g(\vec{x}))) = F$

and $|\det(Dg^{-1})(\vec{y})| = |\det(Dg(\vec{x}))^{-1}|$

$$= \frac{1}{|\det g(\bar{x})|} \quad \text{so}$$

$$(F \circ g^{-1})\ |\det(Dg^{-1})(\bar{y})|$$

$$= F \frac{1}{|\det Dg(\bar{x})|} = f$$

$\square$
