MAT 322/523 Lec 14 Mar 4

Recall: • $S \subset \mathbb{R}^n$ bounded is rectifiable iff 1 integrable on S

• $\mathrm{Vol}(S) = \int_S 1$

• $S \subset \mathbb{R}^n$ is rectifiable $\Leftrightarrow$ S is bounded and $\mu(\partial S) = 0$

Improper integrals (§15 Munkres)

Want to integrate over sets that are not necessarily bounded.

Def: $A \subset \mathbb{R}^n$ open, $f: A \to \mathbb{R}$ continuous.

If $f(\vec{x}) \geq 0$ for $\vec{x} \in A$ we def

$$\boxed{\int_A f := \sup_{\substack{D \text{ compact} \\ \text{rectifiable}}} \int_D f}$$

(if supremum exists)

Def $A \overset{\text{open}}{\subset} \mathbb{R}^n$, $f: A \to \mathbb{R}$ continuous

$$f_+(\vec{x}) := \max(f(\vec{x}), 0)$$

$$f_-(\vec{x}) := \min(-f(\vec{x}), 0)$$

$$\boxed{\int_A f := \int_A f_+ - \int_A f_-}$$

An equivalent formulation:

Thm $A \subset \mathbb{R}^n$ open, $f: A \to \mathbb{R}$ continuous.
$\{C_N\}_{N=1}^{\infty}$ sequence of compact rectifiable subsets: $C_N \subset \operatorname{int} C_{N+1}$ $\forall N$. Then
f integrable $\Leftrightarrow$ sequence $\int_{C_N} |f|$ bounded

$$\int_A f = \lim_{N \to \infty} \int_{C_N} f.$$

Partitions of unity (§16 Munkres)

New approach to improper integrals:

Break up function into pieces instead of breaking up A.

<u>Lma</u>: $Q \subset \mathbb{R}^n$ rectangle. There's a smooth (C^∞) function $\phi: \mathbb{R}^n \to \mathbb{R}$ such that $\phi(\vec{x}) > 0$ for $\vec{x} \in \operatorname{int} Q$

$\phi(\vec{x}) = 0$ otherwise

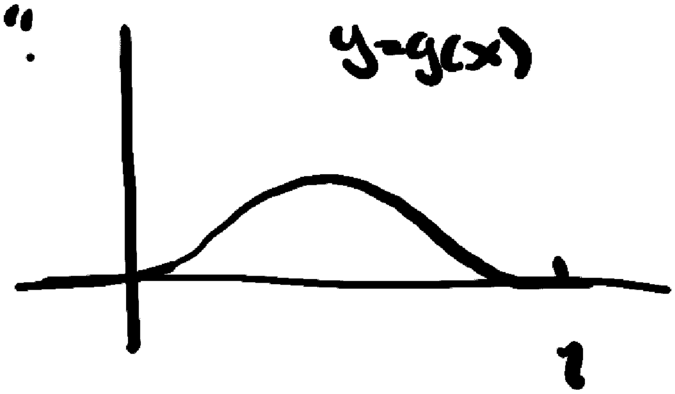
<u>Proof</u>: $Q = \prod_{i=1}^{n} [a_i, b_i]$

$$\phi(\vec{x}) := g\left(\frac{x_1 - a_1}{b_1 - a_1}\right) \cdot \cdots \cdot g\left(\frac{x_n - a_n}{b_n - a_n}\right)$$

where
$$g: \mathbb{R} \to \mathbb{R}, \quad x \longmapsto \begin{cases} e^{-\frac{1}{x(1-x)}} & 0 < x < 1 \\ 0 & \text{else} \end{cases}$$

can be shown to be smooth.

"bump function":



$\square$

Lma: Let $\{A_i\}_{i\in I}$ be open sets, and $A = \bigcup_{i\in I} A_i \overset{\text{open}}{\subset} \mathbb{R}^n$. Then $\exists$ rectengles $\{Q_i\}_{i=1}^{\infty}$, $Q_i \subset A$ such that

(1) $A = \bigcup_{i=1}^{\infty} Q_i$

(2) $\forall i \ \exists j : Q_i \subset A_j$

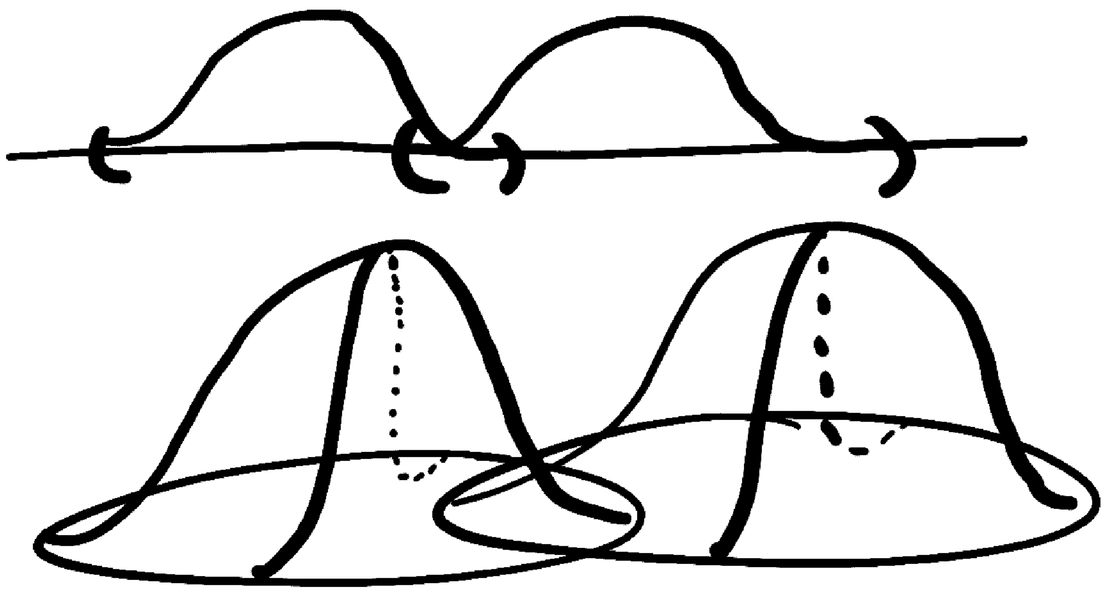
(3) $\vec{x} \in A$ has a nghd $\mathcal{U}$ that only intersects finitely many of the sets.

Can cover an open set w/ countably many rectengles so that no intersection is infinite

Def: The support of a function $\phi : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\operatorname{supp} \phi = \overline{\{\vec{x} \mid \phi(\vec{x}) \neq 0\}}$$

Idea of partition of unity:



collection of bump functions $\phi_1, \dots$ that add up to 1 everywhere.

Def: A partition of unity of an open cover $\{A_i\}_{i \in J}$ of $A \subset \mathbb{R}^n$ is a collection of smooth functions $\{\phi_i : \mathbb{R}^n \to \mathbb{R}\}_{i \in I}$ such that

(1) $\phi(\vec{x}) \geq 0$

(2) $\operatorname{supp} \phi_i \subset A_i \quad \forall i \in I$

(3) $\forall \vec{x} \in A$ there's a nghd that only intersect finitely many of the sets $\operatorname{supp} \phi_i$.

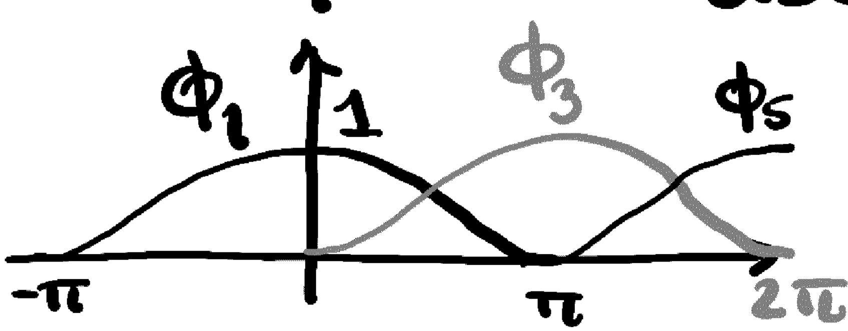
(4) $\sum_{i \in I} \phi_i(\vec{x}) = 1 \quad \forall \vec{x} \in A$

Rmk (3) $\Rightarrow$ Sum in (4) is finite.

Thm: If $A = \bigcup_{i \in I} A \overset{\text{open}}{\subset} \mathbb{R}^n$, then a partition of unity exists.

Ex: $f : \mathbb{R} \to \mathbb{R}$

$$x \longmapsto \begin{cases} \cos^2\left(\frac{x}{2}\right) & -\pi \leq x \leq \pi \\ 0 & \text{else} \end{cases}$$



(f is C^1 but not C^2).

For $m \in \mathbb{Z}_{\geq 0}$ $\phi_{2m+1}(\vec{x}) := f(x - m\pi)$

$m \in \mathbb{Z}_{\geq 1}$ $\phi_{2m}(\vec{x}) := f(x + m\pi)$

Then $\{\phi_i\}_{i=1}^{\infty}$ is a partition of unity:

For $x \in [0, \pi]$

$$\phi_1(\vec{x}) + \phi_3(\vec{x}) = \cos^2\left(\frac{x}{2}\right) + \cos^2\left(\frac{x-\pi}{2}\right)$$

$$= \cos^2\left(\frac{x}{2}\right) + \sin^2\left(\frac{x}{2}\right) = 1.$$

$\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$

Lemma: $A \overset{\text{open}}{\subset} \mathbb{R}^n$, $f: A \to \mathbb{R}$ continuous

If $f = 0$ outside of a compact subset of A, then $\int_A f = \int_C f$.

Thm: $A \overset{\text{open}}{\subset} \mathbb{R}^n$, $f: A \to \mathbb{R}$ continuous

If $\{\phi_i\}_{i=1}^{\infty}$ is a partition of unity s.t. $\operatorname{Supp} \phi_i \subset A$ compact, then

$$\int_A f \text{ exists} \Leftrightarrow \sum_{i=1}^{\infty} \left[\int_A \phi_i |f| \right]$$

converges.

In this case

$$\int_A f = \sum_{i=1}^{\infty} \left[\int_A \phi_i f \right]$$

$$= \sum_{i=1}^{\infty} \left[\int_{\operatorname{supp}\phi_i} \phi_i f \right]$$

Change of variables (§17 Munkres)

Recall single variable calculus:

<u>Thm</u>: $I = [a,b]$, $g: I \to \mathbb{R}$ C^1 function with $g'(x) \neq 0$ for $x \in (a,b)$. Then $g(I) = [g(a), g(b)]$. If $f: g(I) \to \mathbb{R}$ continuous

$$\int_{g(a)}^{g(b)} f = \int_a^b (f \circ g) g'.$$

Ex: $\int_0^1 (2x^2+1)^{10}(4x)$.

Set $g(x) = 2x^2+1$, then $g'(x) = 4x$
$f(y) = y^{10}$. $\neq 0$ in $(0,1)$.

$$\int_0^1 f(g(x)) \cdot g'(x) \overset{\text{Thm}}{=} \int_1^3 y^{10}$$

Def: Let $A \overset{\text{open}}{\subset} \mathbb{R}^n$. A diffeomorphism of $\mathbb{R}^n$ is an injective C^∞-function $g: A \to \mathbb{R}^n$: $Dg(\vec{x}) \neq 0$ $\forall \vec{x} \in A$.

Rmk:
- We might change C^∞ to C^r and keep track of the r.
- $g: A \to \mathbb{R}^n$ is a diffeo. iff $A, B \overset{\text{open}}{\subset} \mathbb{R}^n$: $g: A \to B$ injective & so that g and $g^{-1}: B \to A$ are C^∞.

(This follows from inverse function thm)

Thm. Let $A, B \overset{\text{open}}{\subset} \mathbb{R}^n$ and let $g: A \to B$ be a diffeo. Let $f: B \to \mathbb{R}^n$ be continuous.

Then f integrable iff $(f \circ g)|\det Dg|$ is integrable over A, and

$$\int_B f = \int_A (f \circ g)|\det Dg|$$

Ex: $B = \{(x,y) \mid x, y > 0,\ x^2+y^2 < a^2\}$

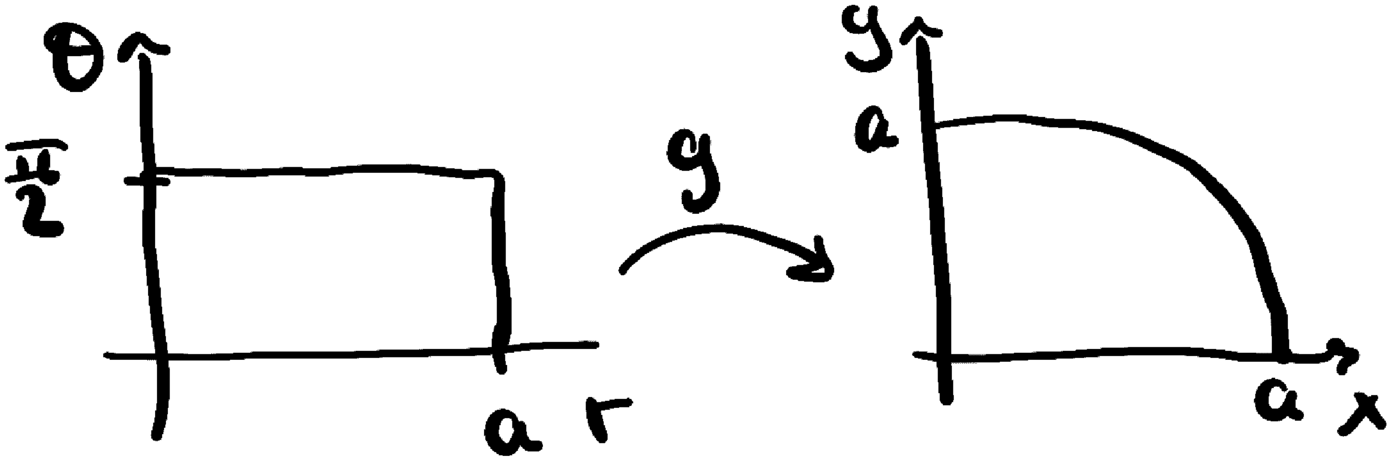
$\int_B x^2y^2$. Polar coords $g: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$g(r,\theta) = (r\cos\theta, r\sin\theta)$$

$$Dg(r,\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}$$

$$\det Dg(r,\theta) = r$$

$$A = \{(r,\theta) \mid 0 < r < a,\ 0 < \theta < \tfrac{\pi}{2}\}$$



Det $Dg \neq 0$, so $g: A \to B$ diffeo.

$$\int_B x^2y^2 = \int_A (r\cos\theta)^2 (r\sin\theta)^2 r$$

$$= \int_A r^5 \cos^2\theta \sin^2\theta$$

Compute via Fubini + another substitution.