MAT 322/523 Lec 11 M Feb 26

<u>Recall</u>: $S \subset \mathbb{R}^n$ bounded, $f, g : S \to \mathbb{R}$ integrable

- $\max(f(\vec{x}), g(\vec{x}))$ and $\min(f(\vec{x}), g(\vec{x}))$ integrable
- $\int_S af + bg = a\int_S f + b\int_S g$
- $f \leq g \Rightarrow \int_S f \leq \int_S g$
- $\left|\int_S f\right| \leq \int_S |f|$
- $T \subset S$, $f(\vec{x}) \geq 0 \Rightarrow \int_T f \leq \int_S f$
- $\int_{T \cup S} f = \int_T f + \int_S f - \int_{T \cap S} f$

From now on we will assume that functions $f : S \to \mathbb{R}$ are continuous. (Not just integrable)

Still an obvious problem:

$$\int_S f = \int_Q f_S \ , \quad f_S(\vec{x}) = \begin{cases} f(\vec{x}), & \vec{x} \in S \\ 0, & \text{else} \end{cases}$$

$Q \supset S$ ← rectangle

Still may be discontinuous at ∂S.

<u>Thm</u>: $S \subset \mathbb{R}^n$ bounded, $f: S \to \mathbb{R}$ bounded, continuous.

$$E := \{ \vec{x}_0 \in \partial S \mid \lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) \neq 0 \}$$

Then
$$\mu(E) = 0 \iff f \text{ integrable}.$$

<u>Proof</u>: $\boxed{\Rightarrow}$

We will show $\vec{x}_0 \in \mathbb{R}^n \setminus E$

$\Rightarrow f_S$ continuous at $\vec{x}_0$.

f_S thus possibly only discontinuous in E, but $\mu(E)=0$ guarantees integrability by earlier result.

There are 3 cases

1. $\vec{x}_0 \in \operatorname{int} S$
2. $\vec{x}_0 \in \operatorname{ext} S$
3. $\vec{x}_0 \in \partial S$

$\boxed{1.}$ Then $f=f_S$ in a neighborhood of $\vec{x}_0$ since $\operatorname{int} S$ is open. so f_S continuous at $\vec{x}_0$ since f is.

$\boxed{2.}$ Similar to the above $f_S=0$ in a nghd of $\vec{x}_0$, so its continuous at $\vec{x}_0$.

$\boxed{3.}$ We may or may not have $\vec{x}_0 \in S$.
By assump. $\vec{x}_0 \notin E$ so $\lim_{\vec{x}\to\vec{x}_0} f(\vec{x}) = 0$
By continuity of f, $f(\vec{x}_0)=0$ if $\vec{x}_0 \in S$. Therefore, since $f_S = f$ or 0

$(\vec{y}\in\mathbb{R}^n)$ $$\lim_{\vec{y}\to\vec{x}_0} f_S(\vec{y}) = 0$$

Hence to prove that f_S is continuous at $\vec{x}_0$ it suffices to show $f_S(\vec{x}_0) = 0$:

If $\bar{x}_0 \notin S$ it follows by def.
Else $f_S(x_0) = f(x_0) = 0$.

$\square$

<u>Thm</u> $S \subset \mathbb{R}^n$ bounded, $f: S \to \mathbb{R}$ bounded, continuous.
If f integrable over S then f integrable over $\operatorname{int} S$ and

$$\int_S f = \int_{\operatorname{int} S} f$$

<u>Rectifiable sets</u> $\boxed{\S 14 \text{ Munkres}}$

Want to extend Fubini's thm to more general sets than rectangles, but all bounded sets is too general.

Def: $S \subset \mathbb{R}^n$ bounded. If the constant function 1 is integrable over S we say that S is **rectifiable**, and we def

$$\operatorname{Vol} S := \int_S 1.$$

Exercise: When $S = \prod_{i=1}^{n} [a_i, b_i]$ is a rectangle, show

$$\int_S 1 = \prod_{i=1}^{n} (b_i - a_i) = \operatorname{Vol} S.$$

Thm: $S \subset \mathbb{R}^n$ is rectifiable $\Longleftrightarrow$ S is bounded and $\mu(\partial S) = 0$.

Proof: $1_S(\vec{x}) = \begin{cases} 1 & \vec{x} \in S \\ 0 & \text{else} \end{cases}$ fails to be continuous only at points on ∂S, so 1_S integrable on $Q \supset S$ iff $\mu(\partial S) = 0$ $\square$

Thm: $S, S_1, S_2 \subset \mathbb{R}^n$ rectifiable

(a) $\operatorname{Vol} S \geq 0$

(b) $S_1 \subset S_2 \Rightarrow \operatorname{Vol} S_1 \leq \operatorname{Vol} S_2$

(c) $S_1 \cup S_2$ and $S_1 \cap S_2$ are rectifiable and
$\operatorname{Vol}(S_1 \cup S_2) = \operatorname{Vol}(S_1) + \operatorname{Vol}(S_2) - \operatorname{Vol}(S_1 \cap S_2)$

(d) $\operatorname{Vol} S = 0 \Leftrightarrow \mu(S) = 0$

(e) $\operatorname{int} S$ is rectifiable and $\operatorname{Vol} S = \operatorname{Vol}(\operatorname{int} S)$

(f) $f : S \to \mathbb{R}$ bounded, continuous then f integrable on S.

Proof: Follows from earlier properties in view of $\operatorname{Vol} S = \int_S 1$. $\square$

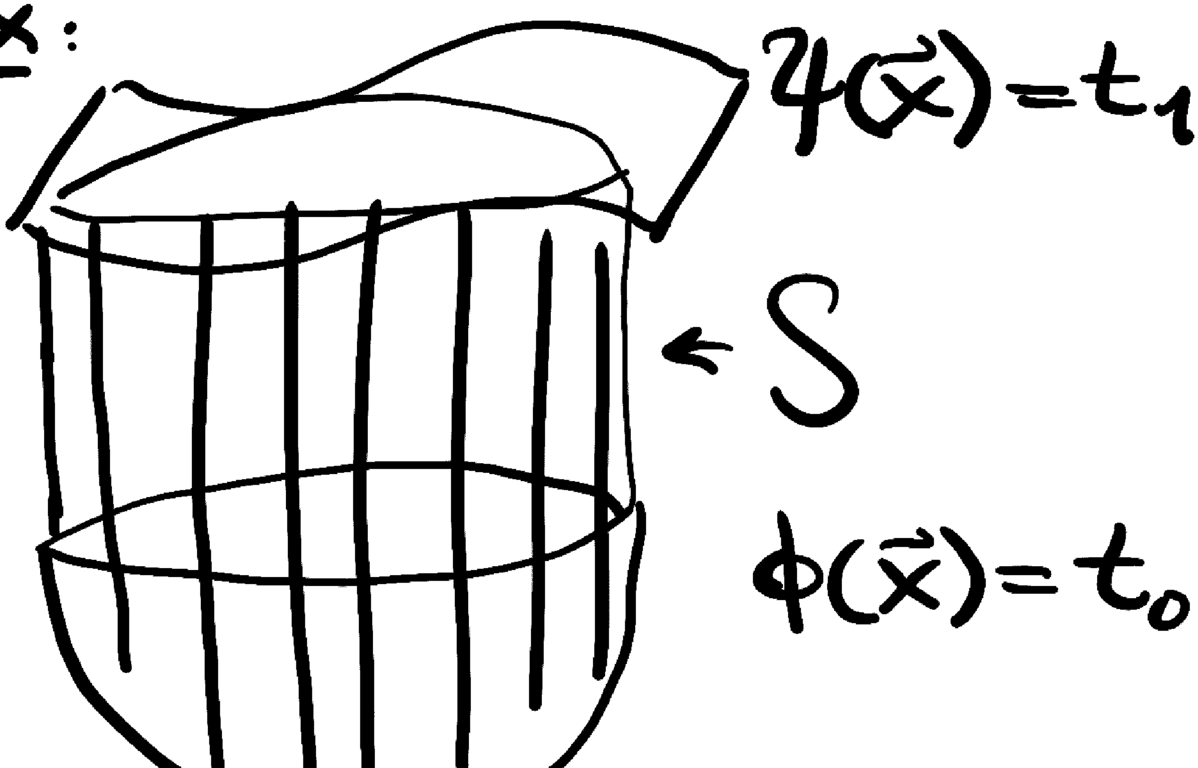
To extend Fubini's thm to more general sets than rectangles, we'll need the following:

<u>Def</u>: $C \subset \mathbb{R}^{n-1}$ compact, rectifiable

$\phi, \psi : C \to \mathbb{R}$ continuous : $\phi(\vec{x}) \leq \psi(\vec{x})$ $\forall \vec{x} \in C$.

$S = \{ (\vec{x}, t) \in C \times \mathbb{R} \mid \phi(\vec{x}) \leq t \leq \psi(\vec{x}) \}$ $\subset \mathbb{R}^n$ is called a <u>simple region</u>.

<u>Ex</u>:



Lemma. Simple regions are compact and rectifiable.

Thm: (Fubini) Simple region

$$S = \{(\vec{x}, t) \in C \times \mathbb{R} \mid \phi(\vec{x}) \le t \le \psi(\vec{x})\}$$

$f: S \to \mathbb{R}$ continuous. Then

$$\int_S f = \int_{\vec{x} \in C} \int_{t=\phi(\vec{x})}^{t=\psi(\vec{x})} f(\vec{x}, t)$$
