MAT 322/523 Lec10 W Feb 21

<u>Recall</u>: Q rectangle, $f: Q \to \mathbb{R}$ bounded

- f integrable $\iff$ f continuous a.e.

- f integrable, $f = 0$ a.e. $\implies$ $\int_Q f = 0$

- f integrable, $\int_Q f = 0$, $f \geq 0$ $\implies$ $f = 0$ a.e.

<u>Integral evaluation</u> (§12 Munkres)

From single variable calculus, the fundamental thm of calculus tells us that we can compute integrals by finding antiderivatives.

Thm (FTC)

(a) $f:[a,b]\to\mathbb{R}$ continuous

then $\frac{d}{dx}\int_a^x f$ exists and is

equal to $f(x)$ for $x\in[a,b]$.

(b) If $f:[a,b]\to\mathbb{R}$ continuous,

and $g'(x)=f(x)$ for $x\in[a,b]$ then

$$\int_a^b f = g(b)-g(a).$$

In multivariable calculus we learn

that if $Q=[a,b]\times[c,d]$ we have

$$\int_Q f = \int_c^d\int_a^b f(x,y)\,dx\,dy.$$

$$= \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy$$

While this is true for f

being continuous, the integral

$\int_{[a,b]} f$ may fail to exist, even

if f is integrable on Q.

<u>Thm</u> (Fubini's theorem)

Let $Q = A \times B$, A rectangle in $\mathbb{R}^k$

B rectangle in $\mathbb{R}^n$

Let $f: Q \to \mathbb{R}$ be bounded, write

$f = f(\vec{x}, \vec{y})$, $(\vec{x}, \vec{y}) \in A \times B$.

If f is integrable, then the two functions of $\vec{x}$

$$\underline{\int}_{\vec{y} \in B} f(\vec{x}, \vec{y}) \text{ and } \overline{\int}_{\vec{y} \in B} f(\vec{x}, \vec{y})$$

are integrable over A, and

$$\int_Q f = \int_{\vec{x}\in A}\left(\underline{\int}_{\vec{y}\in B} f(\vec{x},\vec{y})\right) = \int_{\vec{x}\in A}\left(\overline{\int}_{\vec{y}\in B} f(\vec{x},\vec{y})\right)$$

<u>Cor</u>: If $Q = A\times B$ as above.

If f is integrable on Q and if

$\int_{\vec{y}\in B} f(\vec{x},\vec{y})$ exists for all $\vec{x}\in A$,

then $\int_Q f = \int_{x\in A}\left(\int_{y\in B} f(x,y)\right)$

<u>Proof of Fubini's theorem</u>:

Let $\underline{I}(\vec{x}) := \underline{\int}_{\vec{y}\in B} f(\vec{x},\vec{y})$ for $\vec{x}\in A$.

$\overline{I}(\vec{x}) := \overline{\int}_{\vec{y}\in B} f(\vec{x},\vec{y})$

We will show that $\underline{I}(\vec{x})$ and $\overline{I}(\vec{x})$ are integrable over A, and that their integrals equal $\int_Q f$.

Partitions of Q decompose into products of partitions:

If P is a part of Q then $\exists P_A$ part. of A, P_B part of B such that a subrect R of P is of the form $R = R_A \times R_B$ for subrects R_A, R_B def. by P_A and P_B, respectively.

<u>Step 1</u>: $L(f,P) \le L(\underline{I}, P_A)$

<u>Step 2</u>: $U(f,P) \ge L(\overline{I}, P_A)$

These steps imply

$$
\begin{tikzcd}[column sep=small]
 & U(\underline{I},P_A) \arrow[dr, phantom, "\le"] & \\
L(f,P)\le L(\underline{I},P_A) \arrow[ur, phantom, "\le"] \arrow[dr, phantom, "\le"] & & U(\overline{I},P_A)\le U(f,P) \\
 & L(\overline{I},P_A) \arrow[ur, phantom, "\le"] &
\end{tikzcd}
$$

Step 3: Since f is integrable on Q we can choose P so that

$$U(f,P) - L(f,P) < \varepsilon$$

for any $\varepsilon > 0$. Therefore

$$U(\underline{I}, P_A) - L(\underline{I}, P_A) \leq U(f,P) - L(f,P) < \varepsilon.$$

$\Rightarrow \underline{I}$ integrable on A.

(The case with $\overline{I}$ is similar.)

By def

$$L(\underline{I}, P_A) \leq \int_A \underline{I} \leq U(\underline{I}, P_A), \text{ so}$$

$$\underline{\int_Q} f \leq \int_A \underline{I} \leq \overline{\int_Q} f$$

but since $\underline{\int}_Q f = \overline{\int}_Q f = \int_Q f$ we must have $\int_A \underline{I} = \int_Q f$.

Again, the case for $\overline{I}$ is similar.

<u>Proof of Step 1</u>:

Want to show $L(f,P) \leq L(\underline{I}, P_A)$.

Any subrect R determined by P is of the form $R_A \times R_B$. Letting $\vec{x}_0 \in R_A$, we note

$$m_{R_A \times R_B}(f) \leq m_{R_B}(f(\vec{x}_0, \vec{y}))$$

Let $\vec{x}_0$, and R_A be fixed we get

$$\sum_{R_B} m_{R_A \times R_B}(f) \operatorname{vol}(R_B) \leq L(f(\vec{x}_0, \vec{y}), P_B)$$
$$\leq \underline{\int}_{\vec{y} \in B} f(\vec{x}_0, \vec{y}) = \underline{I}(\vec{x}_0).$$

Since it's true for each $\vec{x}_0 \in R_A$ we get

$$\boxed{\sum_{R_B} m_{R_A \times R_B}(f)\,\mathrm{vol}(R_B) \leq m_{R_A}(\underline{I})}$$

$$L(f,P) = \sum_{R_A \times R_B} m_{R_A \times R_B}(f)\,\mathrm{Vol}(R_A \times R_B)$$

$$= \sum_{R_A} \left[\sum_{R_B} m_{R_A \times R_B}(f)\,\mathrm{vol}(R_B)\right] \mathrm{Vol}(R_A)$$

$$\leq \sum_{R_A} m_{R_A}(\underline{I})\,\mathrm{Vol}(R_A) = L(\underline{I}, P_A).$$

Step 2 is proved similarly. $\square$

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§13 Munkres

We have now only considered integrals over rectangles. It's easy to generalize our def to any bounded set.

<u>Def</u>: $S \subset \mathbb{R}^n$ bounded, $f: S \to \mathbb{R}$ bounded. Def $f_S : \mathbb{R}^n \to \mathbb{R}$ by

$$f_S(\vec{x}) = \begin{cases} f(\vec{x}) \ , & \vec{x} \in S \\ 0 \ , & \text{otherwise} \end{cases}$$

Let Q be a rectangle : $S \subset Q$.

Def

$$\int_S f := \int_Q f_S \quad \text{if it exists.}$$

In this def we <u>made a choice</u> of a rectangle $Q \supset S$. Need to make sure that our def doesn't depend on this choice!

<u>Lma</u>: If Q, Q' are two rectangles and $f: \mathbb{R}^n \to \mathbb{R}$ is bounded so that

$f(\vec{x})=0$ for all $\vec{x} \in \mathbb{R}^n \setminus (Q \cap Q')$.

Then

$$\int_Q f = \int_{Q'} f,$$

and one of the integrals exist iff the other one does.

Some properties of integrals:

Lma: Let $S \subset \mathbb{R}^n$ be bounded, let $f,g : S \to \mathbb{R}$ be continuous a.e.

Then

$$\max_{\vec{x}\in S}(f(\vec{x}), g(\vec{x})) \text{ and } \min_{\vec{x}\in S}(f(\vec{x}), g(\vec{x}))$$

are continuous a.e.

Thm: $f,g: S \to \mathbb{R}$ integrable,

(a) $af+bg$ $(a,b \in \mathbb{R})$ is integrable, and $\int_S (af+bg) = a\int_S f + b\int_S g$.

(b) If $f(\vec{x}) \leq g(\vec{x})$ $\forall \vec{x} \in S$,

$$\int_S f \leq \int_S g$$

(c) $|f|$ is integrable, and $\left|\int_S f\right| \leq \int_S |f|$.

(d) Let $T \subset S$ and assume f integrable on T. If $f(\vec{x}) \geq 0$ $\forall \vec{x} \in S$, then

$$\int_T f \leq \int_S f.$$

(e) $T \subset \mathbb{R}^n$ bounded. If f is integrable over T, then f is integrable over $T \cup S$ and $T \cap S$; furthermore

$$\int_{T \cup S} f = \int_T f + \int_S f - \int_{T \cap S} f$$
