MAT 322/523 Lec 1, M Jan 22

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§ Admin stuff

Class time: MW 2:30–3:50 pm

Grading:
- Homework 20%
- Midterms 20% each
- Final 40%

Text: Munkres – Analysis on Manifolds

Grader: Shuhao Li (Math 2-109)

Me: Johan Asplund (Math 3-116)

Webpage:
math.stonybrook.edu/~jasplund/mat322_spr24

Gradescope: Enroll with code ZW84R6 if you haven't received invitation.

§ Overview of course

Recall results from single and multivariable calculus:

Thm (Fundamental theorem of calculus)

For a differentiable function F on $[a,b]$

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Thm (Fundamental thm of line integrals)

If C is a curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ from A to B and if f is a differentiable scalar function along C, then

$$\oint_C \nabla f \cdot d\vec{r} = f(B) - f(A)$$

Thm (Green's theorem)

$R \subset \mathbb{R}^2$ region w/ bounding curve C oriented counterclockwise.
$\vec{F} = (F_1, F_2)$ vector field continuously differentiable on R, then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \left(\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x} \right) dA$$

<u>Thm</u> (Stokes' thm)

$S \subset \mathbb{R}^3$ oriented surface w/ boundary C. $\vec{F} = (F_1, F_2, F_3)$ vector field cont. diff'able on S, then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$$

<u>Thm</u> (Divergence thm)

$X \subset \mathbb{R}^3$ domain w/ boundary surface S $\vec{F} = (F_1, F_2, F_3)$ vector field cont diff'able on S, then

$$\oiint_S \vec{F} \cdot d\vec{S} = \iiint_X \operatorname{div} \vec{F} \, dV$$

All of these results are special cases of the general Stokes' thm. It says if we have a space M w/ boundary ∂M and F is a differential form cont diff'able on M then

$$\boxed{\int_{\partial M} F = \int_M dF}$$

GOAL: Develop proper language & tools to rigorously prove the general Stokes theorem.

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§ Review of linear algebra

Munkres §1

Read §1 in Munkres.

- Vector space: A set of "vectors"

$\vec{v} + \vec{w}$, $c\vec{v}$, c scalar (real number)

such that (1) $\vec{x}+\vec{y}=\vec{y}+\vec{x}$ (2) $\vec{x}+(\vec{y}+\vec{z})$
$=(\vec{x}+\vec{y})+\vec{z}$

(3) theres a unique zero vector $\vec{0}$ such that $\vec{v}+\vec{0}=\vec{v}$

(4) $\vec{x}+(-1)\cdot\vec{x}=\vec{0}$ (5) $1\cdot\vec{x}=\vec{x}$

(6) $c(d\vec{x})=(cd)\vec{x}$ (7) $(c+d)\vec{x}=c\vec{x}+d\vec{x}$

(8) $c(\vec{x}+\vec{y})=c\vec{x}+c\vec{y}$.

- vectors $\vec{a}_1,\ldots,\vec{a}_m$ <u>span</u> V if for all $\vec{x}$ we have
$$\vec{x}=c_1\vec{a}_1+\cdots+c_m\vec{a}_m$$ <u>for</u> <u>some</u> scalars $c_1,\ldots,c_m$.

- Vectors $\vec{a}_1,\ldots,\vec{a}_m$ are <u>linearly independent</u> if
$$\vec{0}=d_1\vec{a}_1+\cdots+d_m\vec{a}_m$$ implies
$$d_1=\cdots=d_m=0$$

- vectors $\vec{a}_1,\ldots,\vec{a}_m$ is a <u>basis</u> for V if they are lin indep & span V.

- the <u>dimension</u> is # basis vectors in a basis for V.

- an <u>inner product</u> is an assignment of any two vectors $\vec{x}, \vec{y}$ a real number $\langle \vec{x}, \vec{y} \rangle$ such that (1) $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$

(2) $\langle \vec{x}+\vec{y}, \vec{z} \rangle = \langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle$

(3) $\langle c\vec{x}, \vec{y} \rangle = c\langle \vec{x}, \vec{y} \rangle$

(4) $\langle \vec{x}, \vec{x} \rangle > 0$ if $\vec{x} \neq \vec{0}$.

<u>ex</u>: Scalar product on $\mathbb{R}^n$:

$\vec{x} = (x_1, \dots, x_n)$, $\vec{y} = (y_1, \dots, y_n)$

$\langle \vec{x}, \vec{y} \rangle = x_1 y_1 + \cdots + x_n y_n$.

- a <u>norm</u> is an assignment of any vector $\vec{x}$, a real number $\|\vec{x}\|$ such that

(1) $\|\vec{x}\| > 0$ if $\vec{x} \neq \vec{0}$

(2) $\|c\vec{x}\| = |c| \, \|\vec{x}\|$

(3) $\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

If $\langle \vec{x}, \vec{y} \rangle$ is an inner product on V
then $\|\vec{x}\| := \sqrt{\langle \vec{x}, \vec{x} \rangle}$ is a norm.

- **matrices** $\overset{\text{row}}{n} \times \overset{\text{col}}{m}$

cA, $A+B$

- **matrix mult**

$\underset{n \times \underline{m}}{A} \cdot \underset{\underline{m} \times p}{B} = \underset{n \times p}{C}$ defined if

$\#\text{cols}(A) = \#\text{rows}(B)$.

satisfies usual properties

(1) $A \cdot (B \cdot C) = (A \cdot B) \cdot C$

(2) $A \cdot (B + C) = A \cdot B + A \cdot C$

(3) $(A+B) \cdot C = A \cdot C + B \cdot C$

(4) $(cA) \cdot B = c(A \cdot B) = A \cdot (cB)$

c scalar

(5) For each k, there is an

identity matrix $I_k = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$

such that   k 1:s on diagonal,

If A is $n \times m$ 0:s everywhere else

$$I_n \cdot A = A, \quad A \cdot I_m = A$$

$$\boxed{\triangle!\ A \cdot B \neq B \cdot A}$$

- a linear map is a function $T: V \to W$ such that for all $\vec{x}, \vec{y}$ and scalars c:

(1) $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$

(2) $T(c\vec{x}) = cT(\vec{x})$

if $T: V \to W$, $S: W \to X$ are linear maps, then so is $S \circ T: V \to X$

if T invertible, then $T^{-1}: W \to V$ is linear

- if $\dim V = n$, $\dim W = m$ then any linear map $T: V \to W$ is of the form $T(\vec{x}) = A \cdot \vec{y}$, where A is $m \times n$.

- the <u>column rank</u> of a matrix A is the dim of the space spanned by the columns. It's also the dim of the image of the corr linear map
- the <u>row rank</u> is the dim of the space spanned by the rows.

  column rank = row rank = <u>rank</u>

- <u>elementary row operations</u>

  scalar

  (1) Exchange two rows

  (2) Replace row i by (row i) + c(row j) for $i \neq j$

  (3) Multiply row i by a non-zero scalar.

These operations preserve the rank.

- <u>Gauss-Jordan elimination</u>:

Perform row ops until matrix is of <u>echelon form</u>:

<u>ex</u>

$$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \qquad \bigcirc = \text{pivot elements}$$

rank = # pivot elements = 3.

§2 Matrix inversion and determinants

Performing the elementary row operations correspond to multiplication with the <u>elementary matrices</u>

$$(1)\quad E_{ij} := \begin{pmatrix} 1 &&&&&&&& \\ & \ddots &&&&&&& \\ && 1 &&&&&& \\ &&& 0 & \cdots & 1 &&& \\ &&& \vdots && \vdots &&& \\ &&& 1 & \cdots & 0 &&& \\ &&&&&& 1 && \\ &&&&&&& \ddots & \\ &&&&&&&& 1 \end{pmatrix} \begin{array}{l} \\ \\ \\ \text{row } i \\ \\ \text{row } j \\ \\ \\ \\ \end{array}$$

$$(2)\quad E'_{ij,c} := \begin{pmatrix} 1 &&&&&&&& \\ & \ddots &&&&&&& \\ && 1 &&&&&& \\ &&& 1 & \cdots & c &&& \\ &&& \vdots && \vdots &&& \\ &&& 0 & \cdots & 1 &&& \\ &&&&&& 1 && \\ &&&&&&& \ddots & \\ &&&&&&&& 1 \end{pmatrix} \begin{array}{l} \\ \\ \\ \text{row } i \\ \\ \text{row } j \\ \\ \\ \\ \end{array}$$

(3) $$E''_{i,\lambda} := \begin{array}{c} \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \lambda & & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \\ \text{row } i \\ \\ \\ \\ \end{array}$$

We can check that if A is a matrix, then $E_{ij}A$ is the matrix A w/ rows i & j switched. Matrices of type (2) & (3) corresponds similarly to row ops of type (2) and (3).

<u>Thm</u>: If a square matrix has a left inverse and a right inverse, they are unique and equal.

Proof[*] ← DID NOT DISCUSS IN CLASS

Proof: Assume:

B left inverse: $B \cdot A = I_m$

C right inverse: $A \cdot C = I_n$

Want to show $B = C$.

Indeed:

$$C = I_m \cdot C = (B \cdot A) C = B \cdot (A \cdot C) \quad (*)$$

$$= B \cdot I_n = B$$

Next: uniqueness. I.e. if B_1 is another left inverse we must prove $B_1 = B$. Redo (*) w/ B_1 shows

$B = C = B_1$.

Uniqueness of C is similar!

$\square$

Def: If A has both a left and right inverse, then A is said to be invertible. The unique left and right inverse is simply called the inverse A^{-1}.

<u>Thm</u>: A $n \times m$ matrix is invertible if and only if $n = m = \operatorname{rank} A$.

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§Determinants

Assigns to a <u>square</u> matrix A a real number $\det A$.

<u>Thm</u>: Let A be an $n \times n$ matrix.
$\operatorname{rank} A = n \Leftrightarrow \det A \neq 0$.

In other words, a square matrix is invertible iff its determinant is non-zero.