Recall: • $T \in \mathcal{L}(V,W)$ invertible iff T injective and surjective. We call an invertible linear map an isomorphism.

• Two vector spaces V, W are isomorphic $V \cong W$ if there exists an isomorphism $T: V \to W$.

Thm: Let V, W be fin dim v.sp. then $V \cong W \iff \dim V = \dim W$.

Proof. $\Longrightarrow$: If $V \cong W$ then let $T: V \to W$ be an isomorphism; i.e, $\text{null}\, T = \{0\}$ and $\text{range}\, T = W$. Fund thm of linear maps gives

$$\dim V = \dim \text{null}\, T + \dim \text{range}\, T$$
$$= 0 + \dim W = \dim W.$$

$\Longleftarrow$: Choose bases $v_1, \dots, v_n$ and $w_1, \dots, w_n$ for V and W, resp. We know from a previous result (3.4 in textbook) that there's a unique linear map $T: V \to W$ such that $Tv_k = w_k$ $\forall k = 1, \dots, n$. We now show this map is an isomorphism. Since V and W are fin dim of the same dimension, it suffices to prove $\operatorname{null} T = \{0\}$ (i.e. T injective).

For any $v \in V$ write $v = \sum_{i=1}^{n} c_i v_i$, and assume $v \in \operatorname{null} T$. Then

$$Tv = T\left(\sum_{i=1}^{n} c_i v_i\right) = \sum_{i=1}^{n} c_i T v_i = \sum_{i=1}^{n} c_i w_i = 0$$

$(w_1, \dots, w_n)$ is lin indep, so $c_i = 0$ $\forall i$ $\Rightarrow v = 0$ which means $\operatorname{null} T = \{0\}$. $\square$

<u>Prop</u>: Let V and W be fin dim w/ chosen bases $v_1, \dots, v_n$ and $w_1, \dots, w_m$. Then $\mathcal{M}: \mathcal{L}(V, W) \longrightarrow \mathbb{F}^{m,n}$

is an isomorphism.

Proof. Exercise. $\square$

Corollary: If V, W are fin dim,
$\dim \mathcal{L}(V,W) = (\dim V)(\dim W)$

Proof: It follows from $\mathcal{L}(V,W) \cong \mathbb{F}^{m,n}$ & $\dim \mathbb{F}^{m,n} = n \cdot m$. $\square$

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§ Products & quotients (3E)

Def: Suppose $V_1, \dots, V_m$ are vector spaces.
The product of $V_1, \dots, V_m$ is
$V_1 \times \cdots \times V_m = \{(v_1, \dots, v_m) \mid v_i \in V_i \ \forall i\}$.
Addition and scalar mult. are defined componentwise:

- $(v_1, \dots, v_m) + (w_1, \dots, w_m) := (v_1 + w_1, \dots, v_m + w_m)$
- $\lambda(v_1, \dots, v_m) := (\lambda v_1, \dots, \lambda v_m)$

<u>Prop</u>: If $V_1, \dots, V_m$ are vector spaces, then so is $V_1 \times \cdots \times V_m$.

<u>Proof</u>: Exercise. $\square$

<u>Ex</u>: $\mathcal{P}_5(\mathbb{R}) \times \mathbb{R}^3$ is a vector space, and its elements are $(p(x), v)$ where $p(x) \in \mathcal{P}_5(\mathbb{R})$ and $v \in \mathbb{R}^3$. E.g.

$$(5-6x+4x^2, (3,8,7)) \in \mathcal{P}_5(\mathbb{R}) \times \mathbb{R}^3$$
$$(x+9x^5, (2,2,2)) \in \mathcal{P}_5(\mathbb{R}) \times \mathbb{R}^3$$

Their sum is:

$$(5-6x+4x^2, (3,8,7)) + (x+9x^5, (2,2,2))$$
$$= (5-5x+4x^2+9x^5, (5,10,9)) \in \mathcal{P}_5(\mathbb{R}) \times \mathbb{R}^3.$$

<u>Prop</u>: If $V_1, \dots, V_m$ are fin dim vector spaces, then

$$\dim(V_1 \times \cdots \times V_m) = \dim V_1 + \cdots + \dim V_m$$

<u>Proof Sketch</u>: Pick basis $v_1^k, \dots, v_{m_k}^k$ for V_k $\forall k$.

Then $(v_1^1,0,\dots,0),(v_{m_1}^1,0,\dots,0)$
$,(0,v_1^2,0,\dots,0),\dots,(0,\dots,0,v_{m_k}^k)$ is
a list of length $\dim V_1+\dots+\dim V_k$
that one can show is a basis. $\square$

<u>Prop</u>: Suppose $V_1,\dots,V_m\subset V$ are subspaces.
Define a linear map

$$\Gamma: V_1\times\dots\times V_m\longrightarrow V_1+\dots+V_m$$

by $\Gamma(v_1,\dots,v_m)=v_1+\dots+v_m$.
Then $V_1+\dots+V_m$ is a direct sum iff
Γ is injective.

<u>Proof</u>: By def $V_1+\dots+V_m$ is a direct
sum iff $v_1+\dots+v_m=0 \Rightarrow v_i=0\ \forall i$.
But this is equivalent to

$$\Gamma(v_1,\dots,v_m)=0 \Rightarrow v_i=0\ \forall i,$$

which is equiv to null $\Gamma=\{0\}$,
i.e. injectivity of Γ. $\square$

<u>Prop</u>: Suppose V is fin dim, and $V_1, \dots, V_m \subset V$ are subspaces. Then $V_1 + \dots + V_m$ is a direct sum iff $\dim V_1 + \dots + V_m = \dim V_1 + \dots + \dim V_m$.

<u>Proof</u>: The linear map Γ defined above is obviously surjective by def. So the above proposition actually gives:

$$V_1 + \dots + V_m \text{ direct sum} \iff \Gamma \text{ isom.}$$

$$\iff V_1 + \dots + V_m \cong V_1 \times \dots \times V_m$$

$$\iff \dim \sum_{i=1}^{m} V_i = \dim (V_1 \times \dots \times V_m)$$

$$= \sum_{i=1}^{m} \dim V_i$$

$\square$

To define quotient spaces, we need some elementary definitions

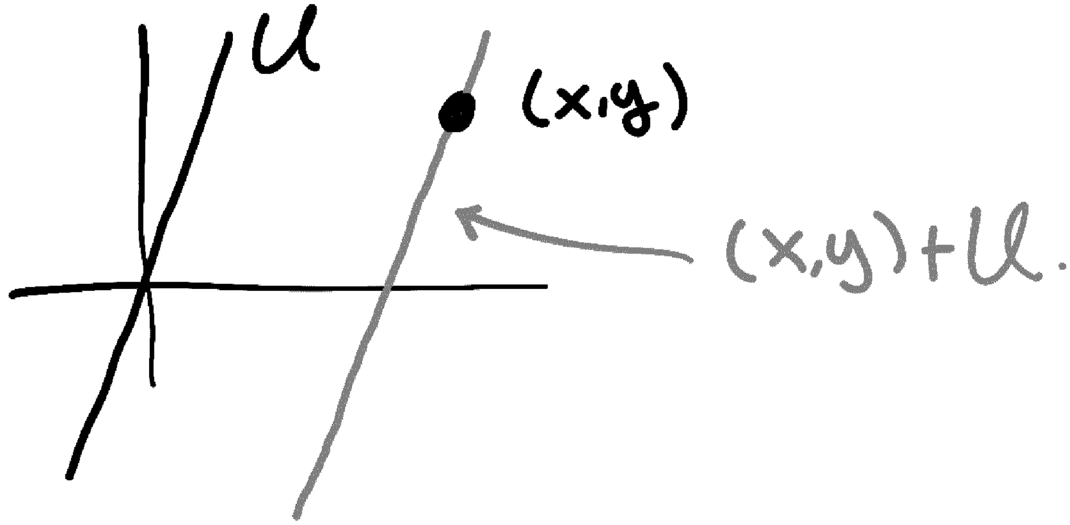
Def: Let $U \subset V$ be a subspace of a vector space, and let $v \in V$. Then define

$$v + U := \{v + u \mid u \in U\}.$$

Ex: Let $V = \mathbb{R}^2$, $U = \{(x,y) \mid y = 2x\}$

line through 0 of slope 2

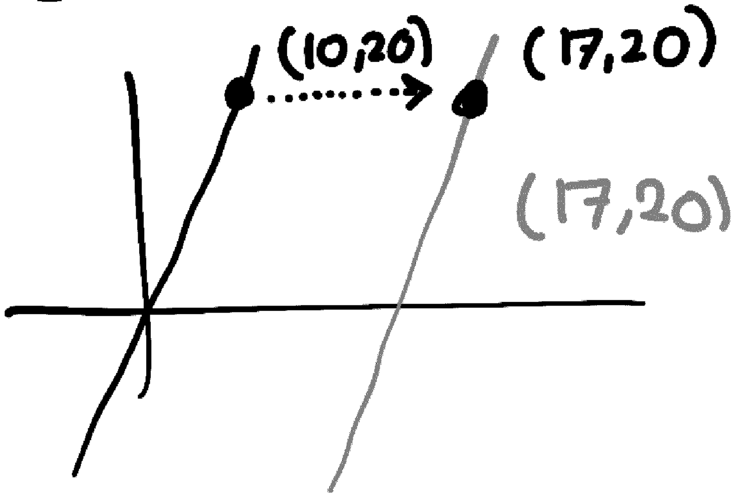
Then for some $(x,y) \in \mathbb{R}^2$, $(x,y) + U$ is the straight line that passes through (x,y).



E.g. $(17,20) + U$ is the line of slope 2 going through $(17,20)$.

We see $(10,20) \in U$ so

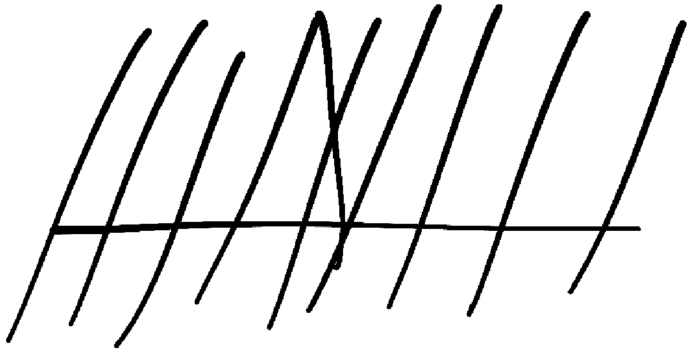
$(17,20)+\mathcal{U}$ is obtained from $\mathcal{U}$ by translating it to the right by 7 units



<u>Def</u>: $v \in V$ and $\mathcal{U} \subset V$, then we call $v+\mathcal{U}$ a <u>translate of $\mathcal{U}$</u>.

<u>Remark</u>: The translate is <u>never</u> a subspace (unless $v=0$, in which case we just have $0+\mathcal{U}=\mathcal{U}$), because $0 \notin v+\mathcal{U}$.

<u>Ex</u>: • If $\mathcal{U}=\{(x,2x) \in \mathbb{R}^2\}$ as above, then all lines in $\mathbb{R}^2$ w/ slope 2 are translates of $\mathcal{U}$



- If $U \subset \mathbb{R}^2$ is a line, then all lines parallel to U are all translates of U.
- If $U \subset \mathbb{R}^3$ is a plane, all planes parallel to U are all translates of U.

<u>Def.</u> Let $U \subset V$ be a subspace. Then

$$V/U = \{\text{all translates of } U\}$$
$$= \{v + U \mid v \in V\}.$$

The goal is to make V/U into a vector space. But we need some propositions first.

<u>Prop</u>: Let $U \subset V$ be a subspace, and $v, w \in V$. Then

$v-w \in U \Leftrightarrow v+U = w+U \Leftrightarrow (v+U) \cap (w+U) \neq \emptyset.$

<u>Proof</u>: To prove the statement we will show the chain of inclusions

$$v-w \in U \overset{①}{\Longrightarrow} v+U = w+U \overset{②}{\Longrightarrow} (v+U)\cap(w+U) \neq \emptyset \overset{③}{\Longrightarrow} v-w \in U$$

$\overset{①}{\Longrightarrow}$: Assume $v-w \in U$. Then we want to prove $v+U = w+U$.

Any element in $v+U$ is of the form $v+u$, for some $u \in U$. Hence

$$v+u = \underbrace{(v-w)}_{\in U} + w + u = w + \underbrace{((v-w)+u)}_{\in U} \in w+U$$

Showing $v+U \subset w+U$. The other inclusion $w+U \subset v+U$ is shown similarly.

$\overset{②}{\Longrightarrow}$: This is obvious.

$\overset{③}{\Longrightarrow}$ Assume $(v+U)\cap(w+U) \neq \emptyset.$

We need to prove $v-w \in U$. By assumption there exists $u_1, u_2 \in U$: $v+u_1 = w+u_2$. But then it means $v-w = u_2 - u_1 \in U$. $\square$

<u>Def</u>: If $U \subset V$ is a subspace, def addition & scalar mult on V/U as follows:

- $(v+U) + (w+U) := (v+w)+U$
- $\lambda(v+U) := (\lambda v)+U$.

<u>Theorem</u>. If $U \subset V$ is a subspace, V/U is a vector space with addition & scalar mult as above.
