MAT 310 L26 Thu May 8

<u>Recall</u>:
- If $T \in \mathcal{L}(V)$ and λ is an eigenvalue, then a non-zero vector is a gen. eigenvector if $(T-\lambda I)^k v = 0$ for some $k \geq 1$
- $T \in \mathcal{L}(V)$ and $\mathbb{F} = \mathbb{C}$ then there's a basis of gen. eigenvectors of T.

<u>Generalized eigenspaces (8B)</u>

<u>Def</u>: Let $T \in \mathcal{L}(V)$ and let $\lambda \in \mathbb{F}$. The gen. eigenspace of T corr to λ is

$$G(\lambda, T) := \{ v \in V \mid (T-\lambda I)^k v = 0 \text{ for some } k \geq 1 \}$$

<u>Prop</u>: Suppose $T \in \mathcal{L}(V)$, $n = \dim V$ and $\lambda \in \mathbb{F}$.

Then $G(\lambda, T) = \operatorname{null}(T-\lambda I)^n$

Proof: If $v \in \text{null}(T-\lambda I)^n$ then $v \in G(\lambda, T)$ by def. For the converse let $v \in G(\lambda, T)$. Then $v \in \text{null}(T-\lambda I)^k$ for some $k \geq 1$. From last time

$$\{0\} = \text{null}(T-\lambda I)^0 \subset \text{null}(T-\lambda I) \subset \text{null}(T-\lambda I)^2 \subset \cdots \subset \text{null}(T-\lambda I)^n = \text{null}(T-\lambda I)^{n+1} = \cdots$$

So we see $v \in \text{null}(T-\lambda I)^n$.
Therefore $G(\lambda, T) = \text{null}(T-\lambda I)^n$. □

Thm: Let $\mathbb{F} = \mathbb{C}$, and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues.

(a) $G(\lambda_k, T)$ is invariant under T for each $k = 1, \ldots, m$

(b) $V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$.

Proof: (a) Previous prop yields $G(\lambda_k, T) = \text{null}(T-\lambda_k I)^n$. So

If $v \in \text{null}(T-\lambda_k I)^n$ we have $(T-\lambda_k I)^n v = 0$. Then

$$(T-\lambda_k I)^n (Tv) = T(T-\lambda_k I)^n v = T(0) = 0$$

So $Tv \in \text{null}(T-\lambda_k I)^n$.

(b) We first show that $G(\lambda_1, T) + \cdots + G(\lambda_m, T)$ is a direct sum. Namely if

$$v_i \in G(\lambda_i, T) \text{ and } v_1 + \cdots + v_m = 0$$

then we must have $v_1 = \cdots = v_m = 0$ since $(v_1, \ldots, v_m)$ are lin dep (see previous lecture).

Picking a basis for V of gen. eigenvectors shows that $v \in V$ can be written as a sum of gen. eigenvectors, finishing the proof. $\square$

Def: Let $T \in \mathcal{L}(V)$. The **multiplicity** of an eigenvalue $\lambda \in \mathbb{F}$ is $\dim G(\lambda, T) = \dim \operatorname{null} (T - \lambda I)^{\dim V}$

Prop: Assume $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. The sum of the multiplicity of all eigenvalues is $\dim V$.

Proof: Follows from

$$V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T). \qquad \square$$

Def: Let $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Let $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues of T w/ multiplicities $(d_1, \dots, d_m)$. The characteristic polynomial of T is:

$$q_T(z) = (z - \lambda_1)^{d_1} \cdots (z - \lambda_m)^{d_m}.$$

Ex: $T: \mathbb{C}^3 \to \mathbb{C}^3$,

$T(x,y,z) = (6x+3y+4z,\ 6y+2z,\ 7z)$.

Wrt the std basis, the matrix is

$$\mathcal{M}(T) = \begin{pmatrix} 6 & 3 & 4 \\ 0 & 6 & 2 \\ 0 & 0 & 7 \end{pmatrix}$$ so we

see that the eigenvalues are 6 and 7, since the matrix is upper triangular.

We may compute $(T-\lambda I)^3$ by taking the cube ob the matrix

$\boxed{\lambda = 6}$

$$\mathcal{M}(T-6I)^3 = \begin{pmatrix} 0 & 3 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}^3 = \begin{pmatrix} 0 & 0 & 10 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

so $(T-6I)^3(x,y,z) = (10z, 2z, z)$ & we see

$G(6,T) = \text{null } (T-6I)^3 = \{(x,y,0)\}$

$\boxed{\lambda = 7}$

$$M(T-7I)^3 = \begin{pmatrix} -1 & 9 & -8 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and}$$

$G(7,T) = \text{null } (T-7I)^3 = \{(0,0,z)\}$.

So mult of $\lambda = 6$ is 2
mult of $\lambda = 7$ is 1.

Characteristic polynomial is

$$q_T(z) = (z-6)^2(z-7).$$

In this case it agrees w/ the minimal polynomial.

<u>Prop</u>: Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$.
Then (a) $\deg q_T = \dim V$
(b) the zeros of q_T are the eigenvalues.

Proof: (a) $\deg q_T = d_1 + \cdots + d_m$
$= \dim G(\lambda_1, T) + \cdots + \dim G(\lambda_m, T)$
$= \dim V$
(b) True by definition. $\square$

Thm (Cayley-Hamilton)
Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$.
$q_T(T) = 0$.

Proof: Distinct eigenvalues are
$\lambda_1, \dots, \lambda_m$ and $d_i = G(\lambda_i, T)$.
It's now a fact that

$$(T - \lambda_i)^{d_i}\big|_{G(\lambda_i, T)} = 0$$

To prove $q_T(T) = 0$ it suffices to
show $q_T(T)\big|_{G(\lambda_i, T)} = 0$ for each
i. By def:

$$q_T(T)\big|_{G(\lambda_i,T)} = (T-\lambda_1 I)^{d_1}\cdots(T-\lambda_m I)^{d_m}\big|_{G(\lambda_i,T)}$$

$$= 0 \qquad \square$$

Prop: Let $\mathbb{F}=\mathbb{C}$ and $T\in\mathcal{L}(V)$ then $p_T \mid q_T$

minimal poly   characteristic poly

Proof: $q_T(T)=0$ and since $\deg p_T \leq \deg q_T$

$$p_T(z) = q_T(z)\,h(z)$$

for some polynomial h. $\square$

Block diagonal matrix:

$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{pmatrix}$$

where each A_i is a square matrix (maybe of different sizes.

Ex
$$\begin{pmatrix} (1) & 0 & 0 & 0 \\ 0 & (2 & 3) & 0 \\ 0 & (0 & 4) & 0 \\ 0 & 0 & 0 & (5) \end{pmatrix} = \begin{pmatrix} A_1 & & O \\ & A_2 & \\ O & & A_3 \end{pmatrix}$$

$$A_1 = (1), \quad A_2 = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}, \quad A_3 = (5).$$

Thm Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Let $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues of T. Then there's a basis of V such that

$$\mathcal{M}(T) = \begin{pmatrix} A_1 & & O \\ & \ddots & \\ O & & A_m \end{pmatrix} \quad \text{where}$$

$$A_k = \begin{pmatrix} \lambda_k & & * \\ & \ddots & \\ O & & \lambda_k \end{pmatrix} \quad \text{is } d_k \times d_k.$$

Proof: $V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$

For each k, one can choose a basis of $G(\lambda_k, T)$ such that the matrix of $(T - \lambda_k I)|_{G(\lambda_k,T)}$ is

$$\begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}.$$

Then the matrix of

$$T|_{G(\lambda_k,T)} = (T - \lambda_k I)|_{G(\lambda_k,T)} + \lambda_k I|_{G(\lambda_k,T)}$$

is $A_k = \begin{pmatrix} \lambda_k & & * \\ & \ddots & \\ 0 & & \lambda_k \end{pmatrix}$. $\square$
