MAT 310 L25 Tue May 6

Generalized eigenvectors (8A)

An operator on a finite dim complex vector spaces may not have a basis of eigenvectors. But if we generalize our definition of eigenvector, it will be true.

Prop: Let $T \in \mathcal{L}(V)$. Then

$$\{0\} = \text{null } T^0 \subset \text{null } T \subset \text{null } T^2 \subset \text{null } T^3 \subset \cdots$$

Proof: For any $k \geq 0$ we have

$$T^{k+1}(v) = T(T^k(v))$$

So if $v \in \text{null } T^k$, then $v \in \text{null } T^{k+1}$

So $\text{null } T^k \subset \text{null } T^{k+1}$. $\square$

Prop: Suppose $T \in \mathcal{L}(V)$ and $m \geq 0$ such that $\text{null } T^m = \text{null } T^{m+1}$ then $\text{null } T^{m+k} = \text{null } T^{m+k+1}$ $k \geq 0$.

Proof: We already know the inclusion null $T^{m+k} \subset$ null T^{m+k+1}. Let $v \in$ null T^{m+k+1}. Then

$$0 = T^{m+k+1}(v) = T^{m+1}(T^k(v))$$

So $T^k(v) \in$ null $T^{m+1} =$ null T^m

So

$$0 = T^m(T^k(v)) = T^{m+k}(v)$$

$\Rightarrow v \in$ null T^{m+k} $\square$

Prop: Suppose $T \in \mathcal{L}(V)$ and $\dim V = n$. Then

$$\text{null } T^n = \text{null } T^{n+1} = \cdots$$

Proof: Already know null $T^n \subset$ null T^{n+1}. Suppose for a contradiction that null $T^n \neq$ null T^{n+1}. We have

$\{0\} =$ null $T^0 \subset$ null $T \subset \cdots \subset$ null T^{n+1}.

Also $\text{null}\, T^n \neq \text{null}\, T^{n+1}$

$\Rightarrow \text{null}\, T^{n-1} \neq \text{null}\, T$

$\vdots$

$\text{null}\, T^0 \neq \text{null}\, T$

So the above chain of inclusions are strict:

$$\{0\} = \text{null}\, T^0 \subsetneq \text{null}\, T \subsetneq \cdots \subsetneq \text{null}\, T^{n+1}$$

So the subspaces must increase in dim:

We get $\dim T^{n+1} > n$

Which is a contradiction, so must have $\text{null}\, T^n = \text{null}\, T^{n+1}$. $\square$

<u>Prop</u>: Suppose $T \in \mathcal{L}(V)$ and $\dim V = n$.

$$V = \text{null}\, T^n \oplus \text{range}\, T^n.$$

<u>Proof</u>: $\text{null}\, T^n + \text{range}\, T^n \subset V$ always exists.

First we show its a direct sum. We will show

$$(\text{null } T^n) \cap (\text{range } T^n) = \{0\}.$$

Namely, let $v \in (\text{null } T^n) \cap (\text{range } T^n)$. Then $T^n v = 0$ and $\exists w \in V$ st $T^n w = v$

Combine them: $0 = T^n v = T^n(T^n w)$ $= T^{2n} w$. Since

$$\text{null } T^n = \text{null } T^{n+1} = \cdots = \text{null } T^{2n}$$

So $T^n w = 0$ & $v = T^n w = 0$.

This shows that null T^n + range T^n is a direct sum.

Next, to prove equality, it suffices to prove that their dimensions agree.

$\dim(\text{null } T^n \oplus \text{range } T^n)$

$= \dim \text{null } T^n \oplus \dim \text{range } T^n$

$= \dim V$ by the fundamental thm for linear maps. Therefore null $T^n \oplus$ range $T^n = V$. $\square$

Def: Let $T \in \mathcal{L}(V)$ and let λ be an eigenvalue of T. A generalized eigenvector is a vector $v \neq 0$ s.t. $(T-\lambda I)^k \neq 0$ for some $k \geq 1$

Thm: Let $\mathbb{F} = \mathbb{C}$, and $T \in \mathcal{L}(V)$. Then there's a basis of V consisting of generalized vector.

Proof: We will prove the theorem by induction on $\dim V = n$.

First, its true for $n=1$ since any $v \neq 0$ is an eigenvector.

Assume $n = p > 1$ and assume

its true for all dimensions $\leq p-1$.
Let λ be an eigenvalue (which exists since $\mathbb{F}=\mathbb{C}$). Then by the previous result

$$V = \text{null}\,(T-\lambda I)^n \oplus \text{range}\,(T-\lambda I)^n$$

First if range $(T-\lambda I)^n = \{0\}$ then $V = \text{null}\,(T-\lambda I)^n$ so any non-zero vector is a generalized vector, and we can pick any basis.
Hence we may assume range $(T-\lambda I)^n \neq \{0\}$

Also $(T-\lambda I)v=0$ so

$(T-\lambda I)^n v = 0 \implies \text{null}(T-\lambda I) \neq \{0\}$.

So we have

$$0 < \dim \text{range}\,(T-\lambda I)^n < n$$

T restricts to a lin op

$$S : \text{range}(T-\lambda I)^n \longrightarrow \text{range}(T-\lambda I)^n$$

By the induction hypothesis we can find a basis $(v_1, \dots, v_m)$ of range$(T-\lambda I)^n$ consisting of gen. eigenvectors.

Extend the basis to $(v_1, \dots, v_n)$ the newly added vectors all lie in null$(T-\lambda I)^n$ so they are gen. eigenvectors too. $\square$

<u>Ex</u>: $T: \mathbb{C}^3 \to \mathbb{C}^3$ s.th.

$$T(x,y,z) = (4y, 0, 5z).$$

We find eigenvalues first:
They are 0 and 5. We find.

$$E(0,T) = \{(x,0,0)\}$$

$$E(5,T) = \{(0,0,z)\}$$

Now since the dimensions doesnt add up to 3 we cant find a basis of eigenvectors

Now $T^3(x,y,z) = T^2(4y,0,5z)$

$= T(0,0,25z) = (0,0,125z)$.

So we find

$E(0,T^3) = \{(x,y,0)\}$

$E(5,T^3) = \{(0,0,z)\}$

So (e_1,e_2,e_3) (std basis) is a basis of generalized eigenvectors.

<u>Prop</u>: Let $T \in \mathcal{L}(V)$. Each gen. eigenvector of T corresponds to <u>only</u> <u>one</u> <u>eigenvalue</u>.

<u>Prop</u>: Let $T \in \mathcal{L}(V)$. Every list of gen. eigenvector of T corr. to distinct eigenvalues, is lin. independent.
