MAT 310 L23 Tue Apr 29

Recall: $T \in \mathcal{L}(V)$

- T self-adjoint $\Leftrightarrow$ $\langle Tv, w\rangle = \langle v, Tw\rangle$ for all $v, w \in V$
- T normal $\Leftrightarrow$ $TT^* = T^*T$.
- Spectral theorem:

$\mathbb{F} = \mathbb{R}$, $\mathbb{C}$

normal

T self-adjoint

$\Leftrightarrow$ there's an ON basis of V of eigenvectors of T

Positive operators (7C)

V = inner product space over $\mathbb{F}$.

Def: A linear operator $T \in \mathcal{L}(V)$ is called positive if T is self-adjoint and $\langle Tv, v\rangle \geq 0$ for all $v \in V$.

Ex: (i) $T: \mathbb{F}^2 \to \mathbb{F}^2$
$$(x,y) \mapsto (x+3y, 3x+9y)$$
Equip $\mathbb{F}^2$ with standard dot product.
$$\mathcal{M}(T) = \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}, \quad \mathcal{M}(T)^* = \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}$$
So T is self-adjoint. Then for $v=(x,y)$ we have
$$Tv \cdot v = (x+3y, 3x+9y) \cdot (x,y)$$
$$= x^2 + 3yx + 3xy + 9y^2$$
$$= (x+3y)^2 \geq 0$$
So T is positive.

Prop: Let $U \subset V$ be any subspace. The orthogonal projection $P_U \in \mathcal{L}(V)$ is positive.

Proof: Write $V = U \oplus U^\perp$. Any $v \in V$ can be written uniquely as
$$v = u + w, \quad u \in U, \quad w \in U^\perp.$$

Then $P_U v = u$. Also let $v' = u' + w'$ for $u' \in U$, $w' \in U^\perp$. Then

$$\langle P_U v, v' \rangle = \langle u, u' + w' \rangle = \langle u, u' \rangle + \underbrace{\langle u, w' \rangle}_{=0} = \langle u, u' \rangle = \langle u, u' \rangle + \underbrace{\langle w, u' \rangle}_{=0}$$

$$= \langle u + w, u' \rangle = \langle v, P_U v' \rangle.$$

This shows that P_U is self-adjoint. Next

$$\langle P_U v, v \rangle = \langle u, u + w \rangle = \langle u, u \rangle$$
$$= \|u\|^2 \geq 0$$

So P_U is positive. $\square$

<u>Def</u>: A linear op $R \in \mathcal{L}(V)$ is called a square root of $T \in \mathcal{L}(V)$ if $R^2 = T$.

<u>EX</u> For a subspace $U \subset V$, the

orthogonal projection is a square root of itself, since $P_U^2 = P_U$.

<u>Ex</u>: $T : \mathbb{F}^3 \to \mathbb{F}^3$, $(x,y,z) = (z,0,0)$

$R : \mathbb{F}^3 \to \mathbb{F}^3$, $(x,y,z) = (y,z,0)$.

Then

$$R^2(x,y,z) = R(y,z,0) = (z,0,0)$$
$$= T(x,y,z)$$

So R is a square root for T.

<u>Thm</u>: Let $T \in \mathcal{L}(V)$. The following are equivalent:

(a) T is positive

(b) T is self-adjoint & all eigenvals of T are non-negative.

(c) There's an ON basis of V s.th. $\mathcal{M}(T)$ is diagonal w/ only non-negative entries

(d) T has a positive square root

(e) T has a self-adjoint square root

(f) $T = R^*R$ for some $R \in \mathcal{L}(V)$.

<u>Proof</u>: We will prove (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (a).

<u>(a) $\Rightarrow$ (b)</u>: By assumption, T is self-adjoint. Its eigenvalues are real, so if $\lambda \in \mathbb{R}$ and $Tv = \lambda v$ for $v \neq 0$, then

$$\langle Tv, v\rangle \geq 0 \Leftrightarrow \langle \lambda v, v\rangle \geq 0$$
$$\Leftrightarrow \lambda \|v\|^2 \geq 0$$
$$\Rightarrow \lambda \geq 0.$$

<u>(b) $\Rightarrow$ (c)</u>: By the spectral thm, there's an ON basis for V s.th. $\mathcal{M}(T)$ is diagonal w/ entries being the eigenvalues. But they are non-negative by assumption.

$\underline{(c) \Rightarrow (d)}$: Pick ON basis for V s.th.

$$\mathcal{M}(T) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, \ \lambda_i \geq 0.$$

Then $\begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{pmatrix}$ is the matrix

for $R \in \mathcal{L}(V)$ s.th. $R^2 = T$. It's clear that R is positive:

It's self-adjoint since

$$\mathcal{M}(R)^* = \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{pmatrix} = \mathcal{M}(R),$$

and for $v = a_1 e_1 + \cdots + a_n e_n$, then

$$\langle Rv, v \rangle = \langle a_1 Re_1 + \cdots + a_n Re_n, a_1 e_1 + \cdots + a_n e_n \rangle$$

$$= \langle a_1 \sqrt{\lambda_1} e_1 + \cdots + a_n \sqrt{\lambda_n} e_n, a_1 e_1 + \cdots + a_n e_n \rangle$$

$$= \langle a_1 \sqrt{\lambda_1} e_1, a_1 e_1 \rangle + \cdots + \langle a_n \sqrt{\lambda_n} e_n, a_n e_n \rangle$$

$$= |a_1|^2 \sqrt{\lambda_1} + \cdots + |a_n|^2 \sqrt{\lambda_n} \geq 0$$

so R is positive.

(d) $\Rightarrow$ (e): Positive implies self-adjoint by definition.

(e) $\Rightarrow$ (f): By assumption $R^2 = T$, and R^*R, so $T = RR = R^*R$.

(f) $\Rightarrow$ (a):

We have $\langle Tv, v\rangle = \langle R^*Rv, v\rangle$
$= \langle Rv, Rv\rangle = \|Rv\|^2 \geq 0$ so T is positive. $\square$

Prop: Every positive operator has a unique positive square root.

If $T \in \mathcal{L}(V)$ is positive, we denote by $\sqrt{T} \in \mathcal{L}(V)$ the positive square root.

Ex: ① Let $I \in \mathcal{L}(\mathbb{F}^n)$. It's clear that $\sqrt{I} = I$. Note that there are other square roots, but they can not be positive.

For instance $T: \mathbb{F}^n \to \mathbb{F}^n$

$$(x_1, \dots, x_n) \mapsto (x_n, \dots, x_1)$$

Then $T^2(x_1, \dots, x_n) = T(x_n, \dots, x_1)$
$= (x_1, \dots, x_n)$.

And: $T(x_1, \dots, x_n) \cdot (x_1, \dots, x_n)$

$= x_1 x_n + x_2 x_{n-1} + \dots + x_n x_1$

does not have to be ≥ 0. Take for instance $(1, 0, \dots, 0, -1)$. Then

$T(1, 0, \dots, 0, -1) \cdot (1, 0, \dots, 0, -1) = -2 < 0$.

Ex: Let $S, T: \mathbb{R}^2 \to \mathbb{R}^2$

$S(x,y) = (x, 2y)$. $T(x,y) = (x+y, x+y)$.

Wrt standard basis of $\mathbb{R}^2$ we have

$$\mathcal{M}(S) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \mathcal{M}(T) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

First, S is clearly positive, since

its diagonal w/ non-negative entries.
The square root of S is

$$\sqrt{S}(x,y) = (x, \sqrt{2}\,y). \quad \mathcal{M}(\sqrt{S}) = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix}$$

To find $\sqrt{T}$ we find an ON basis of eigenvectors.

$$Tv = \lambda v \iff (x+y, x+y) = \lambda(x,y) \iff$$

$$\begin{cases} x+y = \lambda x \\ x+y = \lambda y \end{cases} \iff \begin{cases} (1-\lambda)x + y = 0 \\ x + (1-\lambda)y = 0 \end{cases}$$

One finds $\lambda = 0, 2$ w/ eigenvectors $(1,1)$ and $(1,-1)$. Normalize to get the ON basis $\left(\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\right)$.

Now $\sqrt{T}$ is a lin operator w/ the same eigenvectors but eigenvalues $0, \sqrt{2}$:

$$\sqrt{T}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = (1,1)$$

$$\sqrt{T}\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = (0,0)$$

Then for $(x,y) \in \mathbb{R}^2$ we see

$$(x,y) = \frac{x+y}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) + \frac{x-y}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

so:

$$\sqrt{T}(x,y) = \frac{x+y}{\sqrt{2}}(1,1) = \left(\frac{x+y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}}\right)$$

hence the matrix of $\sqrt{T}$ wrt the standard basis is

$$\mathcal{M}(\sqrt{T}) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$