<u>Recall</u>: $T \in \mathcal{L}(V,W)$

- Adjoint $T^* \in \mathcal{L}(W,V)$ is characterized by $\langle Tv, w\rangle = \langle v, T^*w\rangle$ for all $v \in V$, $w \in W$.
- If $T \in \mathcal{L}(V)$, T is self-adjoint if $T = T^*$
- T is normal if $TT^* = T^*T$.

Spectral theorem (7B)

<u>Prop</u>. If $T \in \mathcal{L}(V)$ is self-adjoint, and $b, c \in \mathbb{R}$ are s.t. $b^2 < 4c$, then $T^2 + bT + cI$ is invertible.

<u>Prop</u>: Let $T \in \mathcal{L}(V)$ be self-adjoint. Then the minimal polynomial of T is $(z-\lambda_1)\cdots(z-\lambda_m)$ for some $\lambda_1, \dots, \lambda_m \in \mathbb{R}$.

Proof. If $\mathbb{F}=\mathbb{C}$ we know

$$p_T(z)=(z-\lambda_1)\cdots(z-\lambda_m)$$ where

$\lambda_1,\dots,\lambda_m\in\mathbb{C}$ are the eigenvalues. But since T is self-adjoint, all eigenvalues are real.

If $\mathbb{F}=\mathbb{R}$ we factor the minimal polynomial as

$$p_T(z)=(z-\lambda_1)\cdots(z-\lambda_m)\cdot(z^2+b_1z+c_1)\cdots(z^2+b_Nz+c_N)$$

for $b_i^2<4c_i$ $\forall i\in\{1,\dots,N\}$.

That it's the minimal polynomial means

$$p_T(T)=(T-\lambda_1I)\cdots(T-\lambda_mI)\cdot(T^2+b_1T+c_1I)\cdots(T^2+b_NT+c_NI)=0 \quad (*)$$

Since T is self-adjoint, we know $T^2+b_iT+c_iI$ is invertible, so we can multiply both sides of (*) by the inverses to get

$$(T-\lambda_1 I)\cdots(T-\lambda_m I)=0.$$

So the minimal poly could not have contained any quadratic factors to begin w/. □

<u>Thm</u> (Real spectral theorem)

Let $\mathbb{F}=\mathbb{R}$ and $T\in\mathcal{L}(V)$. The following are equivalent:

(a) T is self-adjoint

(b) T has a diagonal matrix wrt some ON-basis of V

(c) V has an ON-basis consisting of eigenvectors.

Proof: We will prove (a) ⟺ (b). The equivalence (b) ⟺ (c) follows from 5.55.

(a) ⟹ (b): Assume T is self-adjoint. By previous result, the minimal poly is $(z-\lambda_1)\cdots(z-\lambda_m)$ for some $\lambda_1,\ldots,\lambda_m \in \mathbb{R}$. So 6.37 implies that there's some ON-basis of V such that

$$\mathcal{M}(T) = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_m \end{pmatrix}.$$

But since T is self-adjoint, $T = T^*$, so $\mathcal{M}(T) = \mathcal{M}(T^*) = \mathcal{M}(T)^*$, and

$$\mathcal{M}(T)^* = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ * & & \lambda_m \end{pmatrix} = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_m \end{pmatrix}$$

$$\Longrightarrow \mathcal{M}(T) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{pmatrix}$$ so (b) holds.

(b) $\Rightarrow$ (a) By assumption $\exists$ ON-basis of V s.th.

$$\mathcal{M}(T) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

that satisfies $\mathcal{M}(T) = \mathcal{M}(T)^*$
$= \mathcal{M}(T^*)$

So we must have $T^* = T$. $\square$

Let's now see the complex version:

Thm (Complex spectral thm)

Let $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is normal

(b) T has a diagonal matrix wrt some ON-basis of V

(c) V has an ON-basis consisting of eigenvectors.

Proof: Like before we show (a)$\Leftrightarrow$(b).

(a) $\Longrightarrow$ (b). By 6.38 any cpx lin operator has an upper-triangular matrix wrt some ON-basis.

$$\mathcal{M}(T) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \vdots \\ 0 & & a_{nn} \end{pmatrix}.$$

From this matrix we know

$$Te_k = a_{1k}e_1 + \cdots + a_{kk}e_k \quad \forall k$$

The matrix of the adjoint is

$$\mathcal{M}(T^*) = \begin{pmatrix} \overline{a_{11}} & & 0 \\ \vdots & \ddots & \\ \overline{a_{1n}} & \cdots & \overline{a_{nn}} \end{pmatrix}$$

We get $\|Te_1\|^2 = |a_{11}|^2$ and

$\|T^*e_1\|^2 = |a_{11}|^2 + \cdots + |a_{1n}|^2.$

T being normal implies (7,20) that $\|Te_1\| = \|T^*e_1\|$

$\Rightarrow |a_{12}|^2 + \cdots + |a_{1n}|^2 = 0$

$\Rightarrow a_{12} = \cdots = a_{1n} = 0.$

Repeating this for all e_k finally gives $\mathcal{M}(T) = \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}.$

<u>(b) $\Rightarrow$ (a)</u>: By assumption

$$\mathcal{M}(T) = \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix} \text{ so}$$

$$\mathcal{M}(T^*) = \mathcal{M}(T)^* = \begin{pmatrix} \overline{a_{11}} & & 0 \\ & \ddots & \\ 0 & & \overline{a_{nn}} \end{pmatrix}.$$

Now any diagonal matrices commute, so $TT^* = T^*T$. $\square$

<u>EX</u>: Consider $T \in \mathcal{L}(\mathbb{C}^2)$ def by $T(w,z) = (2w - 3z, 3w + 2z)$. The matrix wrt standard basis is

$\mathcal{M}(T) = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$, and we saw last time that T is a normal operator. We can find eigenspaces:

$$Tv = \lambda v \Leftrightarrow (2w-3z,\ 3w+2z) = (\lambda w, \lambda z).$$

$$\begin{cases} 2w - 3z = \lambda w \\ 3w + 2z = \lambda z \end{cases}$$

$$\Leftrightarrow \begin{cases} (2-\lambda)w = 3z \\ 3w = -(2-\lambda)z \end{cases}$$

$z = \frac{2-\lambda}{3} w$ so

$$3w = -\frac{(2-\lambda)^2}{3} w$$

$$\Leftrightarrow ((2-\lambda)^2 + 9)w = 0$$

($w \neq 0$ since $w = 0 \Rightarrow z = 0$ & we have $(w,z) \neq (0,0)$)

so $(2-\lambda)^2 + 9 = 0$

$\lambda = 2 \pm 3i$.

Can calculate

$$\begin{cases} T(i,1) = (2+3i)(i,1) \\ T(-i,1) = (2-3i)(-i,1) \end{cases}$$

& $(i,1)\cdot(-i,1) = i\cdot\overline{(-i)} + 1\cdot\bar{1} = 0$

So after normalizing (divide each eigenvector by its norm), we get an ON-basis of eigenvectors:

$$\left(\frac{1}{\sqrt{2}}(i,1), \frac{1}{\sqrt{2}}(-i,1)\right).$$