MAT 310 L21 Tue Apr 22

Self-adjoint & normal operators (7A)

We will keep assuming all vector spaces come equipped w/ an inner product.

Def: Suppose $T \in \mathcal{L}(V,W)$. The adjoint of T is the function $T^*: W \to V$ such that

$$\langle Tv, w \rangle = \langle v, T^* w \rangle \quad \forall v \in V,\ w \in W$$

For $T \in \mathcal{L}(V,W)$ and some fixed $w \in W$, we can define a linear functional $\varphi(v) := \langle Tv, w \rangle$. Riesz representation now tells us that there's a unique vector u such that $\varphi(v) = \langle v, u \rangle$. This vector

u is precisely the adjoint:

$T^*w = u.$

<u>Prop</u>: Let $T \in \mathcal{L}(V,W)$, and assume V is fin dim. If $(e_1, \dots, e_n)$ is an ON-basis for V, then

$$T^*w = \overline{\langle Te_1, w\rangle} e_1 + \dots + \overline{\langle Te_n, w\rangle} e_n$$

<u>Proof</u>: This follows from the proof of Riesz representation thm.

<u>Ex</u>: $T : \mathbb{R}^3 \to \mathbb{R}^2$

$$T(x_1, x_2, x_3) = (x_2 + 3x_3, 2x_1)$$

One can either compute T^* using the above expression, or manipulate the expression $\langle Tx, y\rangle = \langle x, T^*y\rangle$.

$$T(x_1, x_2, x_3) \cdot (y_1, y_2) \; (x_2 + 3x_3, 2x_1) \cdot (y_1, y_2)$$

$$= (x_2 + 3x_3) y_1 + 2x_1 y_2$$

$$= x_1 \cdot 2y_2 + x_2 \cdot y_1 + x_3 \cdot 3y_1$$

$= \langle (x_1, x_2, x_3), (2y_2, y_1, 3y_1) \rangle$

So $T^*(y_1, y_2) = (2y_2, y_1, 3y_1)$.

<u>Prop</u>: If $T \in \mathcal{L}(V, W)$ the adjoint $T^*: W \to V$ is linear.

<u>Proof</u>: Let $w_1, w_2 \in W$ and $v \in V$.

$\langle Tv, w_1 + w_2 \rangle = \langle v, T^*(w_1 + w_2) \rangle$. On the other hand:

EQUAL

$\langle Tv, w_1 + w_2 \rangle = \langle Tv, w_1 \rangle + \langle Tv, w_2 \rangle$

$= \langle v, T^* w_1 \rangle + \langle v, T^* w_2 \rangle = \langle v, T^* w_1 + T^* w_2 \rangle$

So $T^*(w_1 + w_2) = T^* w_1 + T^* w_2$.

For $w \in W$, $\lambda \in \mathbb{F}$ we have

$\langle v, T^*(\lambda w) \rangle = \langle Tv, \lambda w \rangle = \bar{\lambda} \langle Tv, w \rangle$

$= \bar{\lambda} \langle v, T^* w \rangle = \langle v, \lambda T^* w \rangle$

So $T^*(\lambda w) = \lambda T^* w$. $\square$

Prop: Let $T \in \mathcal{L}(V,W)$. Then

(a) $(S+T)^* = S^* + T^*$ for any $S \in \mathcal{L}(V,W)$

(b) For any $\lambda \in \mathbb{F}$ $(\lambda T)^* = \bar{\lambda} T^*$

(c) $(T^*)^* = T$

(d) $(ST)^* = T^* S^*$ for $S \in \mathcal{L}(W,U)$

(e) $I^* = I$

(f) If T is invertible, then so is T^*, and $(T^*)^{-1} = (T^{-1})^*$.

Proof: Exercise. It only uses the definition. $\square$

Prop: Let $T \in \mathcal{L}(V,W)$. Then

(a) $\text{null } T^* = (\text{range } T)^\perp$

(b) $\text{range } T^* = (\text{null } T)^\perp$

Proof: (a) $w \in \operatorname{null} T^*$

$\iff T^* w = 0 \iff \langle v, T^* w \rangle = 0 \;\forall v$

$\iff \langle Tv, w \rangle = 0 \;\forall v$

$\iff w \in (\operatorname{range} T)^\perp$. So

$\operatorname{null} T^* = (\operatorname{range} T)^\perp$.

(b) From (a) $\operatorname{null} T^* = (\operatorname{range} T)^\perp$,

So, because $(T^*)^* = T$, we also have $\operatorname{null} T = (\operatorname{range} T^*)^\perp$.

Now take orthogonal complements:

$\operatorname{range} T^* = (\operatorname{null} T)^\perp$. $\square$

Def: Let $A \in \mathbb{F}^{m \times n}$. The conjugate transpose of A is $A^* \in \mathbb{F}^{n \times m}$ w/ entries $(A^*)_{j,k} = \overline{A_{k,j}}$.

Ex: $A = \begin{pmatrix} 2 & i & 1 \\ -1 & -2 & 1-i \end{pmatrix}$

$A^* = \begin{pmatrix} 2 & -1 \\ -i & -2 \\ 1 & 1+i \end{pmatrix}$

Prop: Let $T \in \mathcal{L}(V,W)$. Let $(e_1, \dots, e_n)$ and $(f_1, \dots, f_m)$ be ON-bases for V & W, respectively. Then

$\boxed{\mathcal{M}(T^*) = \mathcal{M}(T)^*}$

matrices wrt the bases $(e_1, \dots, e_n)$, $(f_1, \dots, f_m)$

Def: A linear operator $T \in \mathcal{L}(V)$ is called self-adjoint if $T^* = T$.

Ex: Let $c \in \mathbb{F}$, and

$T: \mathbb{F}^2 \to \mathbb{F}^2,\ (x,y) \mapsto (2x+cy, 3x+7y)$.

Then we can find its adjoint:

$$T(x,y)\bullet(z,w) = (2x+cy,\,3x+7y)\bullet(z,w)$$

$$= (2x+cy)z + (3x+7y)w$$

$$= (2z+3w)x + (cz+7w)y$$

$$= (x,y)\bullet(2z+3w,\ cz+7w)$$

$$T^*(z,w) = (2z+3w,\ cz+7w).$$

We see $T^* = T$ if $\boxed{c=3}$.

<u>Prop</u>: Every eigenvalue of a self-adjoint operator is real.

<u>Proof</u>: Let $T: V \to V$ be self-adjoint

Let $\lambda \in \mathbb{F}$ be an eigenvalue.

$Tv = \lambda v$ for some $v \neq 0$. Then

$$\langle Tv, v\rangle = \langle \lambda v, v\rangle = \lambda\langle v, v\rangle = \lambda\|v\|^2.$$

On the other hand:

$$\langle Tv, v\rangle = \langle v, Tv\rangle = \langle v, \lambda v\rangle = \bar{\lambda}\langle v, v\rangle$$

$= \bar{\lambda} \|v\|^2.$

So $\lambda \|v\|^2 = \bar{\lambda} \|v\|^2 \iff \lambda = \bar{\lambda}$
$\iff \lambda \in \mathbb{R}.$ $\square$

<u>Prop</u>: If $T \in \mathcal{L}(V)$ is self-adjoint, then

$$Tv \perp v \ \forall v \iff T = 0.$$

<u>Ex</u>: The linear operator
$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \ (x,y) \mapsto (-y,x)$
Can not be self-adjoint since

$$T(x,y) \bullet (x,y) = (-y,x) \bullet (x,y) = 0$$

for any $(x,y) \in \mathbb{R}^2$, but $T \neq 0$.
We can also compute the adjoint:

$$T(x,y) \bullet (z,w) = (-y,x) \bullet (z,w)$$
$$= -yz + xw = (x,y) \bullet (w,-z)$$

So $T^* = -T$.

<u>Def</u>: A linear operator $T \in \mathcal{L}(V)$ is <u>normal</u> if $TT^* = T^*T$, i.e., if T commutes with its adjoint.

<u>Prop</u>: If $T \in \mathcal{L}(V)$ is self-adjoint, then it's normal.

<u>Ex</u>: $T: \mathbb{R}^2 \to \mathbb{R}^2$, $(x,y) \mapsto (2x-3y, 3x+2y)$

Then we can compute

$T^*(z,w) = (2z+3w, -3z+2w)$. Their matrices are

$$\mathcal{M}(T) = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \quad \mathcal{M}(T^*) = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}$$

$T \neq T^*$ but we can check

$\mathcal{M}(T)\mathcal{M}(T^*) = \mathcal{M}(T^*)\mathcal{M}(T)$ so

$TT^* = T^*T$

The following is very important:

<u>Prop</u>: Let $T \in \mathcal{L}(V)$ be normal. Then eigenvectors corresponding to distinct eigenvalues are orthogonal.
