MAT 310 L17 Thu Apr 3

<u>Recall</u>: Let V be a vector sp over $\mathbb{F}$

An <u>inner product</u> is a function $V \times V \longrightarrow \mathbb{F}$, $(u,v) \mapsto \langle u,v \rangle$ that satisfies the following properties:

- (Positivity) $\langle v,v \rangle \geq 0 \;\; \forall v \in V$
- (Definiteness) $\langle v,v \rangle = 0 \iff v = 0$
- (Additivity in the first entry)
  $\langle u+v, w \rangle = \langle u,w \rangle + \langle v,w \rangle \;\; \forall u,v,w \in V$
- (Homogeneity in the first entry)
  $\langle \lambda u, v \rangle = \lambda \langle u,v \rangle \;\; \forall \lambda \in \mathbb{F},\, u,v \in V$
- (Conjugate symmetry)
  $\langle u,v \rangle = \overline{\langle v,u \rangle} \;\; \forall u,v \in V$

<u>Ex</u>: • The dot product on $\mathbb{R}^n$.

For x,y def $x \cdot y := x_1y_1 + \cdots + x_ny_n \in \mathbb{R}$.

This is an inner product.

- Let $w, z \in \mathbb{C}^n$. Then
$\langle w, z \rangle = w_1 \overline{z}_1 + \cdots + w_n \overline{z}_n$ is an inner product.

- On $\mathbb{C}^2$, $\langle w, z \rangle = 2 w_1 \overline{z}_1 + w_2 \overline{z}_2$ is an inner product.

- For any $c_1, \ldots, c_n \geq 0$
$\langle w, z \rangle = c_1 w_1 \overline{z}_1 + \cdots + c_n \overline{w_n z_n}$ is an inner product.

- $\langle w, z \rangle = w_1 \overline{z}_1 - w_2 \overline{z}_2$ is <u>not</u> an inner product. Positivity fails.
e.g. $w = z = (0,1)$, then
$$\langle w, z \rangle = 0 \cdot 0 - 1 \cdot 1 = -1.$$

- If $C^0([-1,1])$ is the vector space of continuous real-valued functions on $[-1,1]$, then $\langle f, g \rangle = \int_{-1}^{1} fg \, dx$ is an inner product.

- If $p, q \in \mathcal{P}(\mathbb{F})$ then

$$\langle p,q\rangle = p(0)q(0) + \int_{-1}^{1} p'(x)q(x)\,dx$$

is an inner product

- If $p,q \in \mathcal{P}(\mathbb{F})$,

$$\langle p,q\rangle = \int_0^\infty p(x)q(x)e^{-x}\,dx$$ is

an inner product.

Def: An inner product sp is a vector space V equipped w/ an inner product.

From now on all vector spaces will be inner product spaces. Let's show some basic properties.

Prop: (a) For each $v \in V$, the map $V \to \mathbb{R}$, $u \mapsto \langle u,v\rangle$ is linear.

(b) $\langle 0,v\rangle = 0 \quad \forall v \in V$

(c) $\langle v,0\rangle = 0 \quad \forall v \in V$

(d) $\langle u, v+w\rangle = \langle u,v\rangle + \langle u,w\rangle \quad \forall u,v,w \in V$

(e) $\langle u, \lambda v\rangle = \bar{\lambda}\langle u,v\rangle \ \forall \lambda \in \mathbb{F},\ u,v \in V.$

<u>Proof</u>: (a) This is a consequence of the additivity & homogeneity in the first entry.

(b) Any linear map maps 0 to 0 so this follows from (a).

(c) By symmetry $\bar{0} = \overline{\langle 0,v\rangle} = \langle v,0\rangle$ and $\bar{0} = 0$.

(d) By symmetry again:

$$\langle u, v+w\rangle = \overline{\langle v+w, u\rangle} = \overline{\langle v,u\rangle + \langle w,u\rangle}$$

$$= \overline{\langle v,u\rangle} + \overline{\langle w,u\rangle} = \langle u,v\rangle + \langle u,w\rangle$$

(e) By symmetry again:

$$\langle u, \lambda v\rangle = \overline{\langle \lambda v, u\rangle} = \overline{\lambda\langle v,u\rangle} = \bar{\lambda}\,\overline{\langle v,u\rangle}$$

$$= \bar{\lambda}\langle u,v\rangle. \qquad \square$$

Now, to measure "lengths" in a general inner product sp, we define the notion of a norm in a general inner product sp.

Def: Given $v \in V$, def

$$\|v\| := \sqrt{\langle v, v\rangle}$$

Ex: (a) $z \in \mathbb{F}^n$ (equipped w/ the Euclidean inner product) then

$$\|z\| = \sqrt{|z_1|^2 + \dots + |z_n|^2}$$ is its associated norm.

(b) Let $f \in C^0([-1,1])$ (v. sp. of continuous functions, $[-1,1] \to \mathbb{R}$) w/ inner prod $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$. Then

$$\|f\| = \sqrt{\int_{-1}^{1} f(x)^2\,dx}.$$

Prop: Let $v \in V$.

(a) $\|v\| = 0 \iff v = 0$

(b) $\|\lambda v\| = |\lambda| \cdot \|v\| \quad \forall \lambda \in \mathbb{F}$.

Proof (a) $\|v\|^2 = \langle v, v\rangle = 0 \iff v = 0$

by definiteness of inner product. And $\|v\|^2 = 0 \Leftrightarrow \|v\| = 0$.

(b) $\|\lambda v\|^2 = \langle \lambda v, \lambda v\rangle = \lambda \langle v, \lambda v\rangle$

$$= \lambda \cdot \bar{\lambda} \langle v, v\rangle = |\lambda|^2 \langle v, v\rangle$$

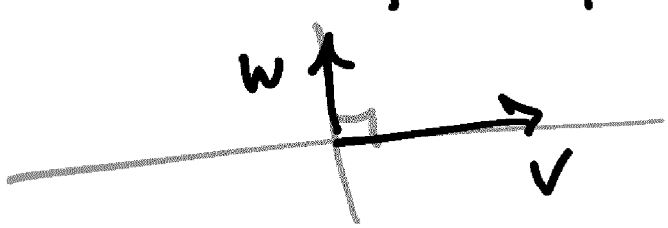
$\Rightarrow \|\lambda v\| = |\lambda| \cdot \|v\|$. $\square$

Inner products can be used to measure angles.

<u>Def</u>: We say that $u, v \in V$ are orthogonal if $\langle u, v\rangle = 0$.

<u>Rmk</u>: $\langle u, v\rangle = 0 \Leftrightarrow \langle v, u\rangle = 0$ so two vectors being orthogonal is a symmetric relation.

<u>Ex</u>: In $\mathbb{R}^2$ two vectors are orthogonal iff the vectors span perpendicular lines



$\langle v, w\rangle = 0$

Def: (a) 0 is orthogonal to every $v \in V$.

(b) 0 is the only vector that's orthogonal to itself.

Proof: This is just a rephrasing of the definitions & properties

$$\langle 0, v \rangle = 0 \quad \text{and} \quad \langle v, v \rangle = 0 \Leftrightarrow v = 0. \qquad \square$$

Prop (Pythagorean theorem)

If $u, v \in V$ are orthogonal, then

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2.$$

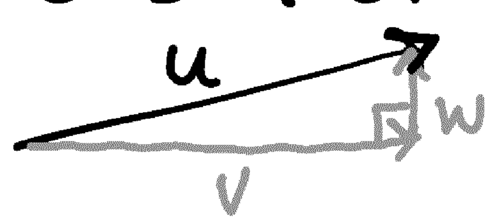
Proof: $\|u+v\|^2 = \langle u+v, u+v \rangle$

$$= \langle u, u+v \rangle + \langle v, u+v \rangle = \langle u, u \rangle + \overbrace{\langle u, v \rangle}^{=0}$$

$$+ \underbrace{\langle v, u \rangle}_{=0} + \langle v, v \rangle = \|u\|^2 + \|v\|^2.$$

$\square$

Given $u \in V$ we'd now like to decompose u as the sum of two orthogonal vectors:



Prop: Suppose $u, v \in V$. Then set $c = \frac{\langle u,v\rangle}{\|v\|^2}$, and $w = u - \frac{\langle u,v\rangle}{\|v\|^2} v$.

Then (a) w is orthogonal to v

(b) $u = cv + w$

Thm (Cauchy-Schwarz inequality)

Let $u, v \in V$. Then

$$|\langle u,v\rangle| \leq \|u\| \, \|v\|.$$

It's an equality iff $u = \lambda v$ for some $\lambda \in \mathbb{F}$.

Proof: Assume $v \neq 0$ (else the ineq clearly holds). Write $u = cv + w$ as in the previous prop (orthogonal decomp) $c = \frac{\langle u,v\rangle}{\|v\|^2}$. Then Pythagorean thm gives

$$\|u\|^2 = \|cv\|^2 + \|w\|^2 = \left\| \frac{\langle u,v\rangle}{\|v\|^2} v \right\|^2 + \|w\|^2$$

$$= \frac{|\langle u,v\rangle|^2}{\|v\|^4}\|v\|^2 + \|w\|^2 = \frac{|\langle u,v\rangle|^2}{\|v\|^2} + \underbrace{\|w\|^2}_{\geq 0}$$

$$\geq \frac{|\langle u,v\rangle|^2}{\|v\|^2} \iff |\langle u,v\rangle|^2 \leq \|u\|^2\|v\|^2 \quad \square$$

Ex: (a) Let $\mathbb{R}^n$ be equipped w/ the Euclidean inner product. Then Cauchy-Schwarz says

$$(x_1y_1+\cdots+x_ny_n)^2 \leq (x_1^2+\cdots+x_n^2)(y_1^2+\cdots+y_n^2).$$

(b) If $f,g \in C^0([-1,1])$ w/ the inner product from before, then

$$\left(\int_{-1}^{1} fg\,dx\right)^2 \leq \left(\int_{-1}^{1} f^2\,dx\right)\left(\int_{-1}^{1} g^2\,dx\right).$$

Thm (Triangle inequality)

Let $u,v \in V$. Then $\|u+v\| \leq \|u\|+\|v\|$ w/ equality iff $u=\lambda v$ for some $\lambda \geq 0$.

Proof: $\|u+v\|^2 = \langle u+v, u+v\rangle$

$= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle$

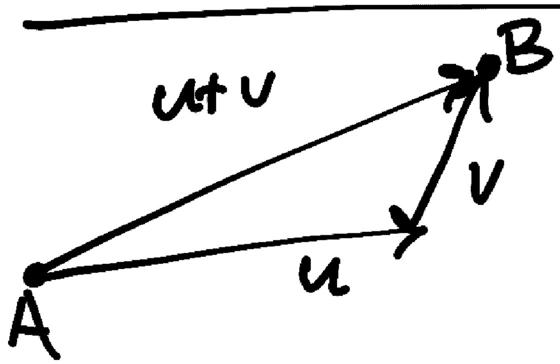
$$= \|u\|^2 + \|v\|^2 + \langle u,v\rangle + \langle v,u\rangle$$

$$= \|u\|^2 + \|v\|^2 + \langle u,v\rangle + \overline{\langle u,v\rangle}$$

$$= \|u\|^2 + \|v\|^2 + 2\,\mathrm{Re}\langle u,v\rangle$$

$$\leq \|u\|^2 + \|v\|^2 + 2\,|\langle u,v\rangle|$$

$$\underset{C-S}{\leq} \|u\|^2 + \|v\|^2 + 2\,\|u\|\,\|v\| = (\|u\| + \|v\|)^2 \quad \square$$



"Straight line from A to B is the shortest path"

<u>Prop</u> (Parallelogram identity)

If $u, v \in V$ then

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$$

<u>Proof</u>: Exercise. Write out $\|u+v\|^2 + \|u-v\|^2$ using the def & simplify. $\square$
