MAT 310 L16 Tue Apr 1

<u>Recall</u>: Let V be fin dim, $T \in \mathcal{L}(V)$.

- T is diagonalizable if there's a basis for V s.t. $\mathcal{M}(T)$ is diagonal.

- T is diagonalizable $\Leftrightarrow$ there's a basis for V consisting of eigenvectors of T.
- If $\dim V = n$ and T has n distinct eigenvalues, then T is diagonalizable.

§ Commuting operators (5E)

<u>Def</u>: Let $T, S \in \mathcal{L}(V)$. We say that T and S <u>commute</u> if $TS = ST$

<u>Def</u>: Let $A, B \in \mathbb{F}^{n,n}$. We say that A and B commute if $AB = BA$.

<u>Ex</u>: • The operator λI for any $\lambda \in \mathbb{F}$ commutes with any $T \in \mathcal{L}(V)$:

$$((\lambda I)T)v = \lambda I(T(v)) = \lambda T(v) = T(\lambda v)$$
$$= (T(\lambda I))v \quad \forall v \in V.$$

- T^n and T^m commute for any $n, m \in \mathbb{N}$.
  In general for any $p, q \in \mathcal{P}(\mathbb{F})$ $p(T)$ and $q(T)$ commute.

- The linear maps
  $T, S : \mathbb{F}^2 \to \mathbb{F}^2$ $\quad T(x,y) = (y,x)$
  $S(x,y) = (x+5y, 3y)$
  do <u>not</u> commute.

It's very rare for two linear maps or matrices to commute.
If you pick two matrices at random the probability that they commute is very close to 0.

Prop: Let $T, S \in \mathcal{L}(V)$ and let $(v_1, \dots, v_n)$ be a basis for V. Then S and T commute iff $\mathcal{M}(S)$ and $\mathcal{M}(T)$ commute.

Proof:

$$ST = TS \Leftrightarrow \mathcal{M}(ST) = \mathcal{M}(TS)$$
$$\Leftrightarrow \mathcal{M}(S)\mathcal{M}(T) = \mathcal{M}(T)\mathcal{M}(S)$$

$\square$

Prop: Let $S, T \in \mathcal{L}(V)$ be two lin ops that commute. Then $E(\lambda, S)$ is invariant under T for all $\lambda \in \mathbb{F}$

Proof: By def $v \in E(\lambda, S) \Leftrightarrow Sv = \lambda v$.

We want to show $Tv \in E(\lambda, S)$

$$S(Tv) = T(Sv) = T(\lambda v) = \lambda(Tv)$$

$\Leftrightarrow Tv \in E(\lambda, S)$. $\square$

The next result is the main reason

we discuss Commuting operators.

<u>Thm</u>: Let $T, S \in \mathcal{L}(V)$ both be diagonalizable. Then $\exists (v_1, \dots, v_n)$ basis for V s.th both $\mathcal{M}(T)$ and $\mathcal{M}(S)$ are diagonal iff $TS = ST$.

<u>Proof</u>: $\Longrightarrow$: If $\mathcal{M}(T)$ and $\mathcal{M}(S)$ are diagonal wrt some basis $(v_1, \dots, v_n)$ of V, then it's clear that $\mathcal{M}(S)\mathcal{M}(T) = \mathcal{M}(T)\mathcal{M}(S)$ since

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{pmatrix} = \begin{pmatrix} \lambda_1\mu_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n\mu_n \end{pmatrix}$$

Therefore $TS = ST$.

$\Longleftarrow$: Now assume $TS = ST$. We will show that we can find a basis for V $(v_1, \dots, v_n)$ s.th. each v_i is an eigenvector for both S & T.

Let S have the distinct eigenvalues $\lambda_1, \dots, \lambda_m$. By diagonalizability of S

we have $V = E(\lambda_1, S) \oplus \cdots \oplus E(\lambda_m, S)$. For each i, $E(\lambda_i, S)$ is invariant under T by the previous prop. Since T is diagonalizable, the lin op.

$$T|_{E(\lambda_i,S)} : E(\lambda_i, S) \longrightarrow E(\lambda_i, S)$$

is diagonalizable for each i. Hence $E(\lambda_i, S)$ admits a basis of eigen-vectors of T. By def such eigenvector is of course also eigenvectors for S by def of $E(\lambda_i, S)$. Assemble this basis together for each i gives the desired basis. $\square$

Prop: If V is a fin dim complex v.sp. and $T, S \in \mathcal{L}(V)$ commute, then there's some $0 \neq v \in V$ that's an eigenvector for both T & S.

Proof: First, by 5.19 S has an

eigenvalue $\lambda \in \mathbb{C}$. Then by def $E(\lambda, S) \neq \{0\}$ so $\exists$ an eigenvector $v \in E(\lambda, S)$. By commutativity of T & S, $E(\lambda, S)$ is invariant under T so $T|_{E(\lambda,S)} : E(\lambda, S) \longrightarrow E(\lambda, S)$ has an eigenvalue & therefore an eigenvector; this is also an eigenvector of S by def. $\square$

<u>Ex</u>: Let $T, S : \mathbb{F}^2 \longrightarrow \mathbb{F}^2$ be def by

$$T(x,y) = (2x, 3y)$$
$$S(x,y) = (5x, 7y).$$

These linear operators obviously commute; their matrices are both diagonal wrt the standard basis:

$$\mathcal{M}(T) = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad \mathcal{M}(S) = \begin{pmatrix} 5 & 0 \\ 0 & 7 \end{pmatrix}.$$

$e_1 = (1,0)$ is an eigenvector for T w/ eigenvalue 2 and eigenvector for S w/ eigenvalue 5. $e_2=(0,1)$ is an eigenvector for T w/ eigenvalue 3 and eigenvector for S w/ eigenvalue 7.

<u>Ex</u>: Let $S,T: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$

$T(x,y)=(y,0)$

$S(x,y)=(x,y)$

Obviously $ST=TS$. So the linear maps must share an eigenvector. The only non-trivial eigenspace of T is $\operatorname{span}((1,0))$

$T(1,0)=(0,0)=0\cdot(1,0)$

$S(1,0)=(1,0)=1\cdot(1,0)$.

The operators share the eigenvector $(1,0)$.

---

§ Inner product spaces (6A)

As always $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$

Any vsp is over $\mathbb{F}$ unless stated otherwise.

The idea of inner products has to do with measuring lengths & angles in vector spaces.

Def: For $x,y \in \mathbb{R}^n$ the dot product of x,y is def by

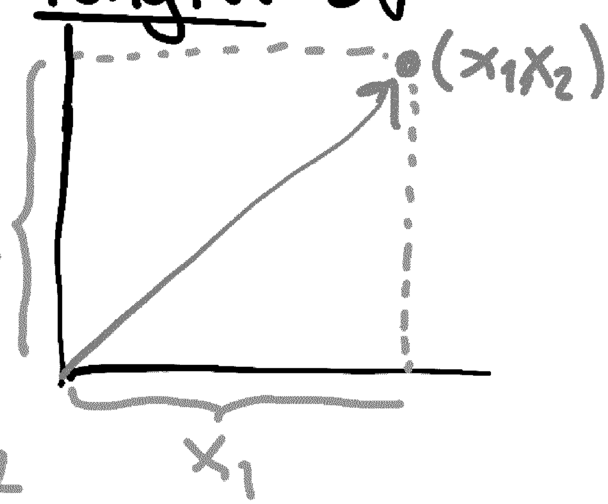
$$x \cdot y := x_1 y_1 + \cdots + x_n y_n.$$

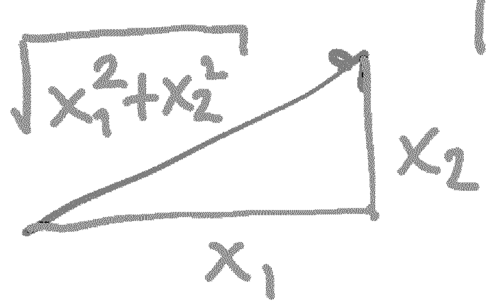
Where $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$.

Note that x, y are vectors & $x \cdot y$ is a real number.

If $x \in \mathbb{R}^n$ the distance to the origin (equivalently the length of the vector x) is

$$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$$





Note $x \cdot x = \|x\|^2$. The dot product satisfies the following properties

- $x \cdot x \geq 0$ for all $x \in \mathbb{R}^n$
- $x \cdot x = 0 \iff x = 0$
- For $y \in \mathbb{R}^n$ fixed, the map
$$\begin{aligned} \mathbb{R}^n &\longrightarrow \mathbb{R} \\ x &\longmapsto x \cdot y \end{aligned}$$
is linear
- $x \cdot y = y \cdot x \quad \forall x, y \in \mathbb{R}^n$!

An inner product is a generalization of the dot product to any vector space.

Recall that if $\lambda \in \mathbb{C}$, $\lambda = a + bi$ then

- $|\lambda| = \sqrt{a^2 + b^2}$
- $\bar{\lambda} = a - bi$
- $|\lambda|^2 = \lambda \bar{\lambda}$.

We will use the notation $\lambda \geq 0$ to mean that λ is real & non-negative. Also, if $r \in \mathbb{R} \subset \mathbb{C}$ then $\bar{r} = r$ since the imaginary part of a real number is zero.

Def: Let V be a vsp over $\mathbb{F}$. An inner product is a function $V \times V \longrightarrow \mathbb{F}$, $(u,v) \mapsto \langle u,v \rangle$ that satisfies the following properties:

- (Positivity) $\langle v,v \rangle \geq 0 \ \forall v \in V$
- (Definiteness) $\langle v,v \rangle = 0 \Leftrightarrow v = 0$
- (Additivity in the first entry)
  $\langle u+v, w \rangle = \langle u,w \rangle + \langle v,w \rangle \ \forall u,v,w \in V$
- (Homogeneity in the first entry)
  $\langle \lambda u, v \rangle = \lambda \langle u,v \rangle \ \forall \lambda \in \mathbb{F},\ u,v \in V$
- (Conjugate symmetry)
  $\langle u,v \rangle = \overline{\langle v,u \rangle} \ \forall u,v \in V$