§ Polynomials (chapter 4)

We will take a brief break from the linear algebra and discuss polynomials. Let us first recall a few basic notions for complex numbers:

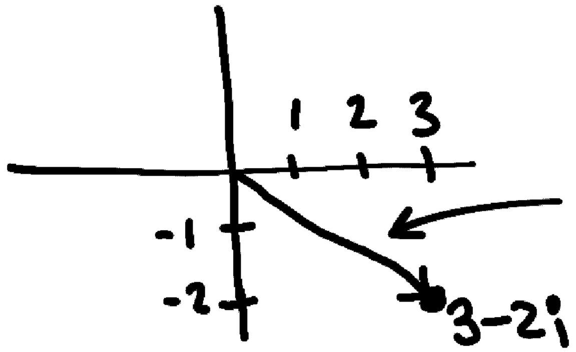
Def: For a complex number
$z = a + bi$

- the real part of z is $\mathrm{Re}(z) := a$
- the imaginary part of z is $\mathrm{Im}(z) := b$.
- The complex conjugate is $\overline{z} := a - bi$
- The absolute value (or modulus) is $|z| = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}$

Ex: $z = 3 - 2i$, then

$\mathrm{Re}(z) = 3$, $\mathrm{Im}(z) = 2$

$\bar{z} = 3 + 2i$, $|z| = \sqrt{3^2 + 2^2} = \sqrt{13}$



length $= |z| = \sqrt{13}$

$|z|$ = length of z viewed as a vector

= distence from $(3,-2)$ to the origin in the plane

Prop: For any $z, w \in \mathbb{C}$

(1) $|z|^2 = z \cdot \bar{z}$

(2) $\overline{z+w} = \bar{z} + \bar{w}$, $\overline{zw} = \bar{z} \cdot \bar{w}$

(3) $\bar{\bar{z}} = z$

(4) $z + \bar{z} = 2\mathrm{Re}(z)$

(5) $|\mathrm{Re}(z)| \leq |z|$, $|\mathrm{Im}(z)| \leq |z|$

(6) $|zw| = |z| \cdot |w|$, $|\bar{z}| = |z|$

(7) $|z+w| \leq |z| + |w|$ (triangle inequality)

Proof: (1)–(6): Exercise

(7) $|z+w|^2 = (z+w)(\overline{z+w}) \overset{(2)}{=} (z+w)(\bar{z}+\bar{w})$

$= z\bar{z} + z\bar{w} + \bar{z}w + w\bar{w}$

$= |z|^2 + |w|^2 + z\bar{w} + \overline{z\bar{w}} = |z|^2 + |w|^2 + 2\mathrm{Re}(z\bar{w})$

$\overset{(5)}{\leq} |z|^2 + |w|^2 + 2|z\bar{w}| = |z|^2 + |w|^2 + 2|z|\cdot|w|$

$= (|z| + |w|)^2$

Taking square roots gives the result. $\square$

Def: A scalar $\lambda \in \mathbb{F}$ is a zero (or root) of $p \in \mathcal{P}(\mathbb{F})$ if $p(\lambda) = 0$.

Prop: Let $p \in \mathcal{P}(\mathbb{F})$ be a polynomial of degree $m \geq 1$, and let $\lambda \in \mathbb{F}$.

Then $p(\lambda) = 0 \iff$ there's a degree $m-1$ polynomial $q \in \mathcal{P}(\mathbb{F})$ such that $p(x) = (x-\lambda)q(x)$.

__Proof:__ $\Longrightarrow$: Assume $p(\lambda)=0$.

Then let $p(x) = a_0 + a_1 x + \cdots + a_m x^m$. Then

$$p(x) = p(x) - p(\lambda)$$

$$= (a_0 + a_1 x + \cdots + a_m x^m) - (a_0 + a_1\lambda + \cdots + a_m\lambda^m)$$

$$= a_1(x-\lambda) + a_2(x^2-\lambda^2) + \cdots + a_m(x^m - \lambda^m).$$

Now, for any k we have

$$x^k - \lambda^k = (x-\lambda)\sum_{j=1}^{k} \lambda^{j-1} x^{k-j}, \quad \text{so}$$

every term in the above polynomial contains a factor of $(x-\lambda)$, so we can factor it out & get

$$p(x) = (x-\lambda)q(x).$$

$\Longleftarrow$: If $p(x) = (x-\lambda)q(x)$ then obviously $p(\lambda) = \underbrace{(\lambda-\lambda)}_{=0} q(\lambda) = 0$. $\square$

__Prop:__ $p \in \mathcal{P}(\mathbb{F})$ of degree m has at most m zeros.

Proof: We prove it by induction on the degree m, with base case $m=0$.

If $m=0$, p is a non-zero constant. So it has no zeros, which shows that the prop is true in this case.

Assume the prop is true for all $m \leq r$ for some r. Then we need to prove it's true for $r+1$.

Consider p of degree $r+1$. If it has no zeros we are done. Else if there's a zero $\lambda \in \mathbb{F}$ we write

$$p(x) = (x-\lambda) q(x)$$

where $\deg q = r$. By the induction hypothesis q has at most r roots, so $p(x)$ has at most $r+1$ roots. $\square$

Polynomials can be divided.

Prop: Let $p, s \in \mathcal{P}(\mathbb{F})$ with $s \neq 0$. Then there are unique polynomials $q, r \in \mathcal{P}(\mathbb{F})$ such that

$$p = qs + r \text{ and } \deg r < \deg s.$$

"remainder"

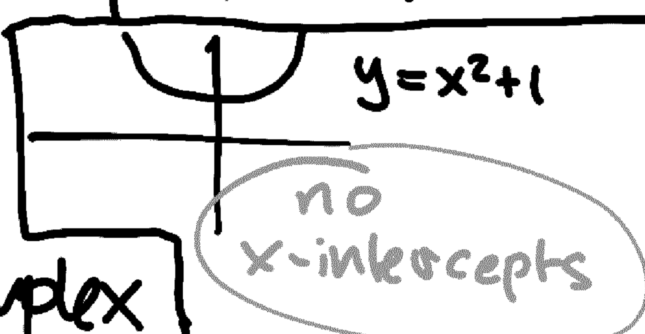
"quotient"

Let us now discuss differences between real and complex polynomials.

Theorem (Fundamental theorem of algebra)

Every non-constant polynomial in $\mathcal{P}(\mathbb{C})$ has a zero in $\mathbb{C}$.

Remark: This is of course <u>not</u> true for real polynomials. $p(x) = x^2 + 1$ has no real solution.



This difference between real & complex

polynomials is what is responsible for the difference between real & complex vector spaces, as we will see later.

Prop: If $p \in \mathcal{P}(\mathbb{C})$ is non-constant, then p has a unique factorization

$$p(x) = c\,(x-\lambda_1)\cdots(x-\lambda_m)$$

where $c, \lambda_1, \ldots, \lambda_m \in \mathbb{C}$.

Proof: If p is non-constant, it has a zero by the fundamental thm of algebra, so we write

$$p(x) = (x-\lambda_1)\,q(x) \text{ where } \lambda_1 \in \mathbb{C}$$

is the zero.

Now if $q(x)$ is constant we're done, else we iterate this:

$q(x)$ non-const mean it has a zero λ_2 so $p(x) = (x-\lambda_1)(x-\lambda_2)\,r(x)$

etc, until we reach

$$p(x) = (x-\lambda_1)\cdots(x-\lambda_m)\,c$$

Constant polynomial

$\square$

<u>Prop</u>: If p is a polynomial with real coefficients & $\lambda \in \mathbb{C}$ is a zero for p, then so is $\bar{\lambda}$.

<u>Proof</u>: λ is a zero:

$$p(\lambda) = a_0 + a_1\lambda + \cdots + a_m\lambda^m = 0.$$

Take complex conjugates to get

$$0 = \overline{a_0 + a_1\lambda + \cdots + a_m\lambda^m} = \bar{a}_0 + \bar{a}_1\bar{\lambda} + \cdots + \bar{a}_m\bar{\lambda}^m$$

$$= a_0 + a_1\bar{\lambda} + \cdots + a_m\bar{\lambda}^m$$

$\square$

<u>Prop</u>: Let $b, c \in \mathbb{R}$. We have

$$x^2 + bx + c = (x-\lambda_1)(x-\lambda_2)$$

for $\lambda_1, \lambda_2 \in \mathbb{R}$ iff $b^2 \geq 4c$.

Proof: The abc-formula gives that the zeros of x^2+bx+c are $x = -\frac{b}{2} \pm \sqrt{(\frac{b}{2})^2 - c}$ which are real iff $(\frac{b}{2})^2 - c \geq 0 \iff b^2 - 4c \geq 0$

This gives two real zeros λ_1, λ_2 & hence we may factor x^2+bx+c as described earlier. □

Even though not every real polynomial has a zero, we can still break real polynomials into reasonably small pieces.

Prop: Let $p \in \mathcal{P}(\mathbb{R})$ be non-constant. Then p has a unique factorization

$$p(x) = c(x-\lambda_1)\cdots(x-\lambda_m) \cdot (x^2+b_1x+c_1)\cdots(x^2+b_Mx+c_M)$$

where $c, \lambda_1, \ldots, \lambda_m, b_1, c_1, \ldots, b_M, c_M \in \mathbb{R}$

and $b_k^2 < 4c_k \;\forall k$.