Name: _____ ID: _____

Answer each question completely. You must justify your answers to get credit. Even a correct answer with no justification will get no credits. Each problem is worth 5 points.

1. Consider the (2×2)-matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, and consider the basis $B = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)$ of $\mathbb{R}^2$. Find the matrix $[A]_B$. Is A diagonalizable?

Solution. Set $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Then we have $[A]_B = S^{-1}AS$. So we need to find the inverse of S first. We can compute $S^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Then

$$[A]_B = S^{-1}AS = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 3 & -1 \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}.$$

□

2. Consider the (3×3)-matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ and consider the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Verify that $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are eigenvectors of A. What are their respective eigenvalues?

Solution. We compute

$$A\vec{v}_1 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = 3\vec{v}_1$$
$$A\vec{v}_2 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = (-1)\vec{v}_2$$
$$A\vec{v}_2 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} = 4\vec{v}_3.$$

This shows that all of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are eigenvectors, and their respective eigenvalues are 3, -1, and 4. □