Name: _____ ID: _____

Answer each question completely. You must justify your answers to get credit. Even a correct answer with no justification will get no credits. Each problem is worth 5 points.

1. Consider the two vectors $\vec{u}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}$. They form an ON-basis for a subspace V in $\mathbb{R}^3$ (you do not need to explain or show this). Find the matrix for the linear transformation $\text{proj}_V \colon \mathbb{R}^3 \to \mathbb{R}^3$.

Solution. Let $Q = \begin{bmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & 0\end{bmatrix}$. Then the matrix is given by

$$QQ^T = \begin{bmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\end{bmatrix} = \begin{bmatrix}\frac{5}{6} & -\frac{1}{6} & \frac{1}{3}\\ -\frac{1}{6} & \frac{5}{6} & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3}\end{bmatrix} = \frac{1}{6}\begin{bmatrix}5 & -1 & 2\\ -1 & 5 & 2\\ 2 & 2 & 2\end{bmatrix}.$$

□

2. For any two vectors $\vec{v} = \begin{bmatrix}v_1\\v_2\end{bmatrix}$ and $\vec{u} = \begin{bmatrix}u_1\\u_2\end{bmatrix}$ in $\mathbb{R}^2$, define $\vec{v} \diamond \vec{u} = v_2u_1 + v_1u_2$.

(a) Show that $\diamond$ is symmetric, meaning that $\vec{v} \diamond \vec{u} = \vec{u} \diamond \vec{v}$.

Solution. We have

$$\vec{v} \diamond \vec{u} = \begin{bmatrix}v_1\\v_2\end{bmatrix} \diamond \begin{bmatrix}u_1\\u_2\end{bmatrix} = v_2u_1 + v_1u_2$$

$$\vec{u} \diamond \vec{v} = \begin{bmatrix}u_1\\u_2\end{bmatrix} \diamond \begin{bmatrix}v_1\\v_2\end{bmatrix} = u_2v_1 + u_1v_2$$

and we see that these expressions are equal. □

(b) Show that $\diamond$ is *not* positive definite by finding a vector $\vec{v}$ such that $\vec{v} \diamond \vec{v} < 0$. Conclude that $\diamond$ is not an inner product on $\mathbb{R}^2$.

Solution. Take for instance $\vec{v} = \begin{bmatrix}1\\-1\end{bmatrix}$. Then

$$\vec{v} \diamond \vec{v} = \begin{bmatrix}1\\-1\end{bmatrix} \diamond \begin{bmatrix}1\\-1\end{bmatrix} = -1 - 1 = -2 < 0.$$

□