MAT 211 L9 Wed Sep 23

Recall:

- A subspace V of $\mathbb{R}^n$ is a collection of vectors such that
  - (1) V contains $\vec{0}$
  - (2) if V contains $\vec{v}_1$ and $\vec{v}_2$, it also contains $\vec{v}_1 + \vec{v}_2$.
  - (3) if V contains $\vec{v}$, it contains $k\vec{v}$ for any scalar k.

- Let $\vec{v}_1, \dots, \vec{v}_m$ be vectors.

- $\vec{v}_i$ is <u>redundant</u> if

  $\vec{v}_i = c_1\vec{v}_1 + \dots + c_{i-1}\vec{v}_{i-1}$ for some scalars $c_1, \dots, c_{i-1}$.

- $\vec{v}_1, \dots, \vec{v}_m$ are <u>linearly independent</u> if there is no redundant vector

- they are <u>linearly dependent</u> if there is at least one redundant

Vector:

- $\vec{v}_1, \dots, \vec{v}_m$ is a basis for V if they span V & are linearly independent.

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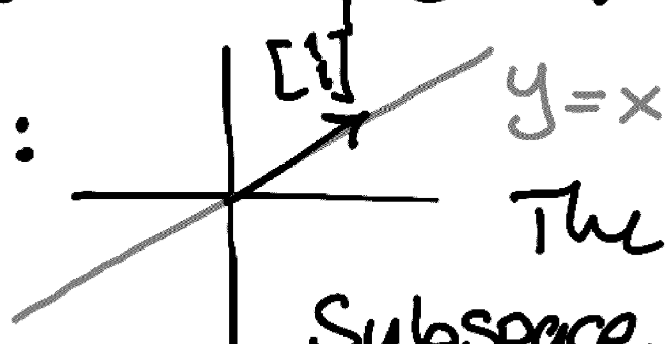
§3.3 Dimension

If V is a subspace of $\mathbb{R}^n$, and $\vec{v}_1, \dots, \vec{v}_m$ is a basis, then any vector $\vec{v}$ in V can be written as a linear combination of the vectors $\vec{v}_1, \dots, \vec{v}_m$ in a unique way. This means

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_m\vec{v}_m$$

has a unique solution $\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$.

Ex:



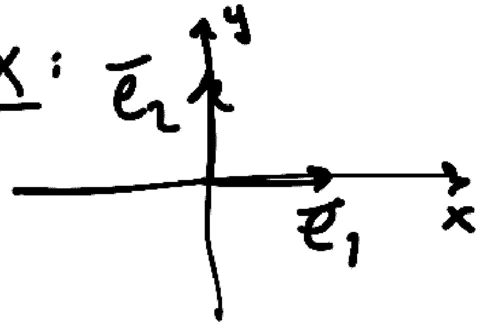
The line is a subspace, and we can

express the line as $\operatorname{span}\left(\begin{bmatrix}1\\1\end{bmatrix}\right)$

$\begin{bmatrix}1\\1\end{bmatrix}$ is a basis.

Any vector in the line is of the form $\begin{bmatrix}k\\k\end{bmatrix} = k\begin{bmatrix}1\\1\end{bmatrix}$,
intuitively this subspace is 1-dimensional.

Ex:



$\vec{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$ Standard vectors in $\mathbb{R}^2$. They span $\mathbb{R}^2$ because $\mathbb{R}^2 = \operatorname{Span}\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right)$. Any vector $\begin{bmatrix}x\\y\end{bmatrix}$ can be written as

$$x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix}.$$

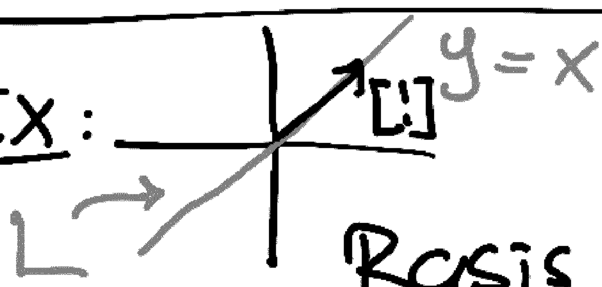
The two vectors $\vec{e}_1$ and $\vec{e}_2$ are also linearly independent because they aren't parallel.

So $\vec{e}_1, \vec{e}_2$ is a basis for $\mathbb{R}^2$.

<u>Theorem</u>: If V is a subspace of $\mathbb{R}^n$, then V has a basis. Moreover, any basis has the same number of vectors.

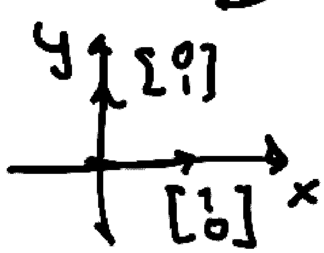
<u>Definition</u>: If V is a subspace of $\mathbb{R}^n$, then the <u>dimension</u> of V is defined as

$$\dim V \overset{\text{def}}{=\!=} \#\text{basis vectors in any basis for } V.$$

<u>Ex</u>: 

$\dim L = 1$

Basis is $\begin{bmatrix}1\\1\end{bmatrix}$

<u>Ex</u>: 

$\vec{e}_1, \vec{e}_2$ is a basis for $\mathbb{R}^2$

$\dim \mathbb{R}^2 = 2$.

Ex: Similarly, in $\mathbb{R}^n$,

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

forms a basis for $\mathbb{R}^n$, so $\dim \mathbb{R}^n = n$.

Theorem: Let V be a subspace of $\mathbb{R}^n$ with $\dim V = m$.

(1) We can find at most m linearly independent vectors in V.

(2) We need at least m vectors to span V.

(3) If m vectors in V are linearly independent, they form a basis of V.

(4) If m vectors in V span V, they form a basis of V.

$\underline{\text{Ex}}$: Consider $A = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{bmatrix}$

Let's find both $\ker A$ and $\operatorname{im} A$.

We first compute $\operatorname{rref}(A)$:

$$\begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{bmatrix} \quad +1 \quad -4 \quad -3$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & -3 & 12 & -15 \\ 0 & 0 & -5 & 20 & 25 \end{bmatrix} \quad +3 \quad +5$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad -2$$

leading

$$\begin{array}{c} \begin{bmatrix} 1 & 2 & 0 & 3 & -4 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \mathrm{rref}(A). \\ \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \end{array}$$

Now to find ker A we need to solve $A\vec{x} = \vec{0}$.

The columns corresponding to non-leading elements need to be assigned parameters.

Set $\begin{cases} x_5 = t \\ x_4 = s \\ x_2 = r \end{cases}$ t, s, r parameters.

$$x_3 = 4x_4 - 5x_5 = 4s - 5t$$

$$x_1 = -2x_2 - 3x_4 + 4x_5$$

$$= -2r - 3s + 4t$$

So $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2r-3s+4t \\ r \\ 4s-5t \\ s \\ t \end{bmatrix}$

$= r\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}.$

Therefore

$\ker A = \operatorname{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}\right)$

This will in fact automaticdly give that $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}$ forms a basis for $\ker A$. So

$\dim(\ker A) = 3.$

Now for the image,

im A = Span of columns in rref(A) containing the leading elements

$= \text{Span}\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right)$

& $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$ forms a basis, so

$\dim(\text{im } A) = 2.$

We can always find a basis for im A by picking out the columns of rref(A) containing the leading elements. So in fact

$$\boxed{\dim(\text{im}(A)) = \text{rank } A}$$

Rank-nullity theorem

For any $(n \times m)$-matrix A, we have

$$\boxed{\dim(\ker A) + \dim(\text{im } A) = m}$$

Sometimes dim ker A is called the "nullity of A", so then the equality is

$$\boxed{(\text{nullity of } A) + (\text{rank of } A) = n}$$