MAT 211 L8 Mon Sep 21

Recall: • $\vec{v}_1, \ldots, \vec{v}_m$ vectors

$\mathrm{Span}(\vec{v}_1, \ldots, \vec{v}_m) = \{$linear combinations of the vectors $\vec{v}_1, \ldots, \vec{v}_m\}$
$= \{c_1\vec{v}_1 + \cdots + c_m\vec{v}_m \mid c_1, \ldots, c_m \text{ scalars}\}$

• $T: \mathbb{R}^m \to \mathbb{R}^n$, $T(\vec{x}) = A\vec{x}$ linear transformation. If

$$A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}$$ then

$$\boxed{\begin{aligned} \mathrm{im}\, T &= \mathrm{span}(\vec{v}_1, \ldots, \vec{v}_m) \\ \ker T &= \{\vec{x} \mid A\vec{x} = \vec{0}\} \end{aligned}}$$

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§3.2 Subspaces

We saw last time that im T and ker T contains $\vec{0}$, are closed under addition & closed under scalar multiplication.

Def: A subset $W \subset \mathbb{R}^n$ is called a subspace if

① W contains the zero vector $\vec{0}$

② W is closed under addition, meaning: if W contains $\vec{w}_1$ and $\vec{w}_2$, then it also contains $\vec{w}_1 + \vec{w}_2$

③ W is closed under scalar multiplication, meaning: if W contains $\vec{w}$ then it contains $k\vec{w}$ for any scalar k.

Last time we saw that for a linear transformation

$T: \mathbb{R}^m \to \mathbb{R}^n$,

- $\text{im}\, T$ is a subspace of $\mathbb{R}^n$
- $\ker T$ is a subspace of $\mathbb{R}^m$

Example: Consider the set

$W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \text{ in } \mathbb{R}^2 \mid x \geq 0 \text{ and } y \geq 0 \right\}$

Is this a subspace?

① W conteins $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ✓

② If W conteins $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ then it means $x_1, x_2 \geq 0$ and $y_1, y_2 \geq 0$.

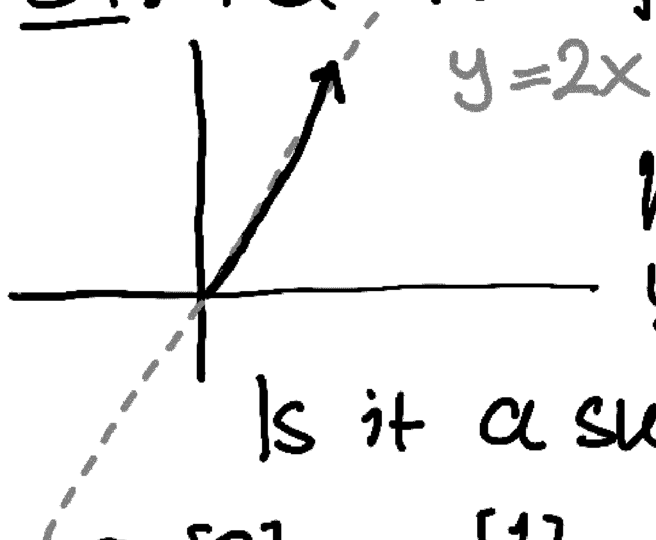
So now $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$.

Since $x_1 + x_2 \geq 0$ and $y_1 + y_2 \geq 0$, then the sum also belongs to W. ✓

③ If W conteins $\begin{bmatrix} x \\ y \end{bmatrix}$ and k is any scalar, does $k\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ ky \end{bmatrix}$ belong to W? In other words, is $kx \geq 0$ and $ky \geq 0$? The answer is <u>no</u>, because we may choose $k < 0$.

So W is <u>not</u> a subspace.

Ex: Let $W = \text{Span}\left(\begin{bmatrix}1\\2\end{bmatrix}\right)$ in $\mathbb{R}^2$



W is the line $y=2x$.

Is it a subspace?

① $\begin{bmatrix}0\\0\end{bmatrix} = 0 \cdot \begin{bmatrix}1\\2\end{bmatrix}$, so W contains $\begin{bmatrix}0\\0\end{bmatrix}$ ✓

② If W contains $\vec{w}_1$ and $\vec{w}_2$, then $\vec{w}_1 = k_1\begin{bmatrix}1\\2\end{bmatrix}$ and $\vec{w}_2 = k_2\begin{bmatrix}1\\2\end{bmatrix}$ for some scalars k_1, k_2. Then $\vec{w}_1 + \vec{w}_2$ $= k_1\begin{bmatrix}1\\2\end{bmatrix} + k_2\begin{bmatrix}1\\2\end{bmatrix} = \underbrace{(k_1+k_2)}_{\text{scalar}}\begin{bmatrix}1\\2\end{bmatrix}$ so W contains $\vec{w}_1 + \vec{w}_2$. ✓

③ If W contains $\vec{w} = k\begin{bmatrix}1\\2\end{bmatrix}$, and ℓ is a scalar, then $\ell\vec{w} = \underbrace{\ell k}_{\text{scalar}}\begin{bmatrix}1\\2\end{bmatrix}$, so W contains $\ell\vec{w}$. ✓

Conclusion is that W is a subspace.

Ex: $W = \{\vec{0}\}$.

① W contains $\vec{0}$ ✓

② $\vec{0}+\vec{0}=\vec{0}$, so W is closed under addition

③ $k\vec{0}=\vec{0}$ for any scalar k, so W is closed under scalar multiplication.

$\Longrightarrow$ W is a subspace.

Note that we never specified what size $\vec{0}$ is, so W is a subspace in any $\mathbb{R}^n$.

The only subspaces of $\mathbb{R}^2$ are: $\{\vec{0}\}$, $\mathbb{R}^2$, any line through the origin.

It's a fun exercise to prove that the span of any collection of vectors is a subspace.

Bases & linear dependence:

Ex: Consider $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$.

Recall by our characterization of the image that

$$\operatorname{im} A = \operatorname{Span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\2\\2\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\3\\4\end{bmatrix}\right)$$

Can we describe the image with fewer vectors than 4?

Note $\begin{bmatrix}2\\2\\2\end{bmatrix} = 2 \cdot \begin{bmatrix}1\\1\\1\end{bmatrix}$ so

$$\operatorname{Span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\2\\2\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\3\\4\end{bmatrix}\right) = \operatorname{Span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\3\\4\end{bmatrix}\right)$$

It's less obvious, but note

$\begin{bmatrix}2\\3\\4\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}1\\1\\1\end{bmatrix}$, so the last vector is also redundant:

$$\operatorname{Span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\3\\4\end{bmatrix}\right) = \operatorname{Span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}\right)$$

Since these vectors are not parallel, we can not reduce the span anymore.

Let $\vec{v}_1, \dots, \vec{v}_m$ be vectors in $\mathbb{R}^n$.

(a) We say $\vec{v}_j$ is <u>redundant</u> in the list $\vec{v}_1, \dots, \vec{v}_m$ if

$\vec{v}_j$ is a linear combination of the vectors $\vec{v}_1, \dots, \vec{v}_{j-1}$.

(b) $\vec{v}_1, \dots, \vec{v}_m$ are <u>linearly independent</u> if none of them are redundant. If at least one vector is redundant, they are <u>linearly dependent</u>

(c) $\vec{v}_1, \dots, \vec{v}_m$ in a subspace V of $\mathbb{R}^n$ form a <u>basis</u> if they are linearly independent, and if they span V.

<u>Ex</u>: Consider $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$ again.

$\text{im } A = \text{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\2\\2\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\3\\4\end{bmatrix}\right)$ we found two redendant vectors: $\begin{bmatrix}2\\2\\2\end{bmatrix}$ and $\begin{bmatrix}2\\3\\4\end{bmatrix}$, which means

$$\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\2\\2\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\3\\4\end{bmatrix} \text{ are } \underline{\text{linearly dependent}}$$

After removing the redendant vectors

$$\begin{aligned}\text{im } A &= \text{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\2\\2\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\3\\4\end{bmatrix}\right)\\ &= \text{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}\right)\end{aligned}$$

We found that $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}$ is a basis for im A.

Generally, to find a basis for im$T$, we first list the columns of the matrix and then find and remove the redendant vectors.

In the previous example we were "lucky" when we found the redundant vectors.

Ex: $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$. Let's find a basis for $\operatorname{im} A = \operatorname{Span}\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix}\right)$.

Need to find redundant vectors.

$\begin{bmatrix}4\\5\\6\end{bmatrix}$ is not a multiple of $\begin{bmatrix}1\\2\\3\end{bmatrix}$.

Is $\begin{bmatrix}7\\8\\9\end{bmatrix}$ a linear combination of $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}$?

Can we write $\begin{bmatrix}7\\8\\9\end{bmatrix} = c_1\begin{bmatrix}1\\2\\3\end{bmatrix} + c_2\begin{bmatrix}4\\5\\6\end{bmatrix}$ for some scalars c_1, c_2? This is a system of linear equations that we can solve:

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix}7\\8\\9\end{bmatrix} ; \quad \left[\begin{array}{cc|c} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{array}\right] \begin{array}{l} -2 \quad -3 \\ \leftarrow \\ \leftarrow \end{array}$$

$$\longrightarrow \left[\begin{array}{cc|c} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{array}\right] \begin{array}{c} \\ -2 \\ \, \end{array} \longrightarrow \left[\begin{array}{cc|c} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{array}\right] \begin{array}{c} \\ \cdot -\frac{1}{3} \\ \, \end{array}$$

$$\longrightarrow \left[\begin{array}{cc|c} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right] \begin{array}{c} \\ -4 \\ \, \end{array} \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

$$\Longrightarrow c_1 = -1, \; c_2 = 2.$$

So $\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = -\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and

$$\operatorname{im} A = \operatorname{Span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}\right) = \operatorname{Span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\right)$$

The idea of a basis of a subspace W is that any vector in W can be expressed in a unique way as a linear combination of its basis vectors.