MAT 211 L7 Wed Sep 16

Recall: • Inverse of a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$

$T(\vec{x}) = A\vec{x}$ is $T^{-1}: \mathbb{R}^n \to \mathbb{R}^n$

$T^{-1}(T(\vec{x})) = \vec{x}$, $T(T^{-1}(\vec{y})) = \vec{y}$

$T^{-1}(\vec{y}) = A^{-1}\vec{y}$

• A is an $(n \times n)$-matrix, then if A^{-1} exists, it's found by:

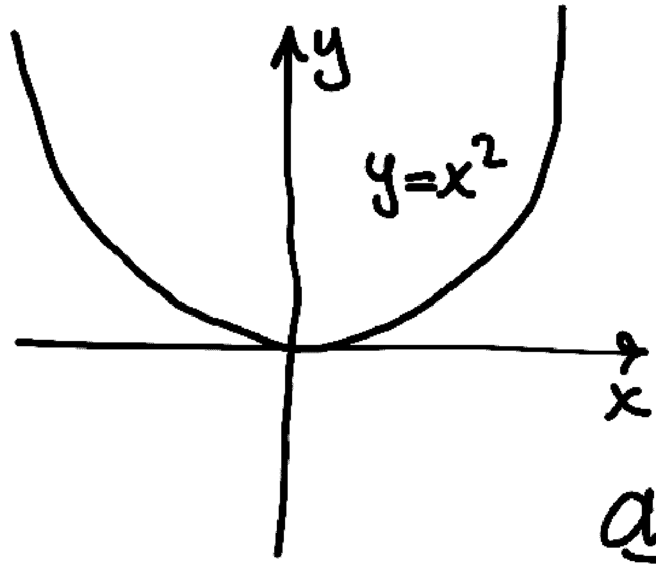
$$[A \mid I_n] \xrightarrow{\text{row ops}} [I_n \mid A^{-1}].$$

§3.1 Image and kernel of linear transformations

Calculus: Consider the function

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

domain range



While the <u>range</u> (all allowed outputs) is $\mathbb{R}$, this function will not actually be <u>able</u> to attain all values in the range. $f(x)=x^2$ is a square, so its non-negative. Only non-negative values of $\mathbb{R}$ are actually "hit" by the function.

$\text{image}(f) = [0,\infty)$.

<u>Def</u>: $T: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ function, then

$\text{image}(T) = \{$ all values in $\mathbb{R}^n$ of the form $T(\vec{x})$ for some $\vec{x}$ in $\mathbb{R}^m \}$.

Ex: Let $T: \mathbb{R} \to \mathbb{R}^2$ be given by $T(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} t$.

What does the image look like?

Any vector in $\mathbb{R}^2$ that is parallel to $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is in the image of T.

For example $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in the image of T.

Ex: $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(\vec{x}) = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \vec{x}$

If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ then

$$T(\vec{x}) = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} x_1 + \begin{bmatrix} 3 \\ 6 \end{bmatrix} x_2 \quad (*)$$

We note that $\begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ So

$$(*) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 2 \end{bmatrix} 3x_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \underbrace{(x_1 + 3x_2)}_{\text{scalar}}$$

So the image of T is all vectors parallel to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Ex: $T: \mathbb{R}^2 \to \mathbb{R}^3$, $T(\vec{x}) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \vec{x}$

Write $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then

$$T(\vec{x}) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} x_2.$$

The two vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ are not parallel, so this cannot be simplified any further.

The image of T is any linear combination of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

If $\vec{v}_1, \dots, \vec{v}_m$ are vectors in $\mathbb{R}^n$, the set of all linear combinations of the vectors $\vec{v}_1, \dots, \vec{v}_m$ is called their <u>span</u>.

$$\text{Span}(\vec{v}_1, \dots, \vec{v}_m) = \{ c_1\vec{v}_1 + \dots + c_m\vec{v}_m \text{ for all scalars } c_1, \dots, c_m \}$$

Generally:

If $T: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ linear transformation, $T(\vec{x}) = A\vec{x}$, then the image of T, denoted $\text{im}(T)$, (or $\text{im}(A)$), is defined as:

$\text{im}(T) =$ Span of columns of A.

Ex ① $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(\vec{x}) = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \vec{x}$

$\text{im}(T) = \text{Span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix}\right) = \text{Span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$

Since $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ is parallel to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

② $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $T(\vec{x}) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \vec{x}$

$\text{im}(T) = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$

The image of a linear transformation satisfies a bunch of properties:

$T: \mathbb{R}^m \longrightarrow \mathbb{R}^n$, $T(\vec{x}) = A\vec{x}$

A is some $(n \times m)$-matrix. Then

(a) $\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ always belongs to $\text{im}(T)$

(b) $\text{im}(T)$ is "closed under addition": If $\vec{v}_1$ and $\vec{v}_2$ are in $\text{im}(T)$, then so is $\vec{v}_1 + \vec{v}_2$

(c) $\text{im}(T)$ is "closed under scalar multiplication": If $\vec{v}$ is in $\text{im}(T)$, then so is $k\vec{v}$ for any scalar k.

These properties will be important later.

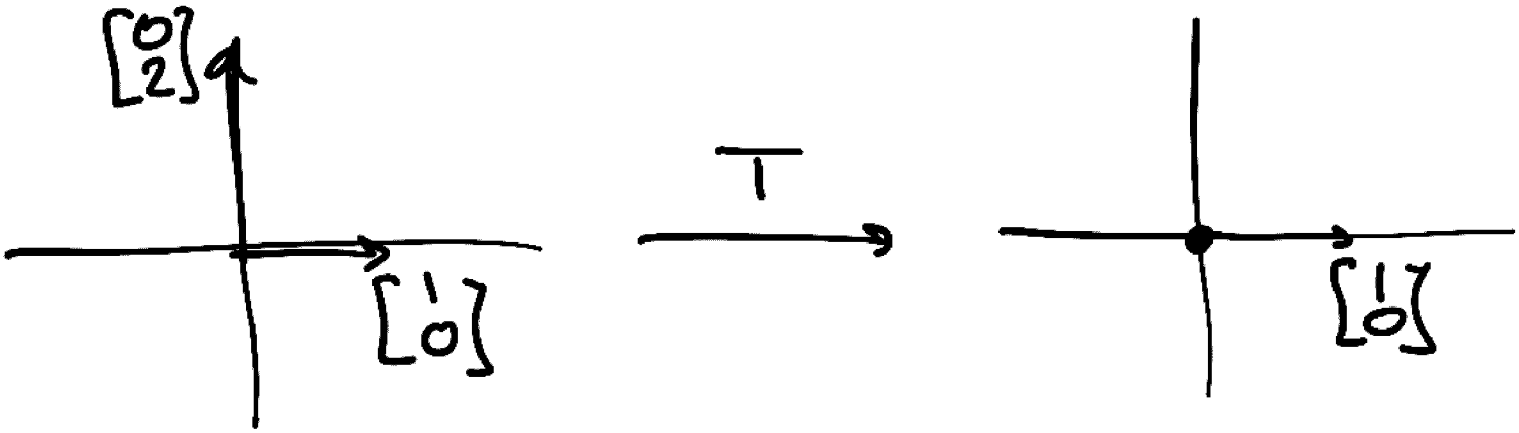
Now we consider another "space of vectors" associated to a linear transformation.

<u>Def.</u> Let $T: \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation $T(\vec{x}) = A\vec{x}$ for some

$(n \times m)$-matrix. The __kernel__ of T is all vectors $\vec{x}$ so that $T(\vec{x}) = \vec{0}$. In other words $\boxed{A\vec{x} = \vec{0}}$.

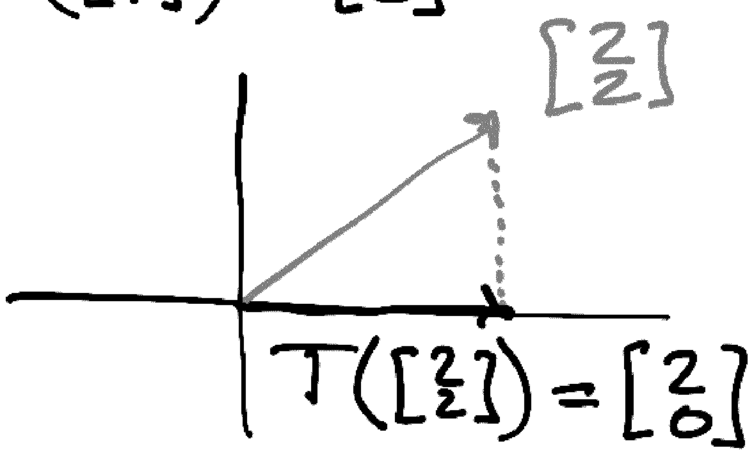
We denote it $\ker(T)$ (or $\ker(A)$).

__Ex__: Consider $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by orthogonal projection onto the x-axis.

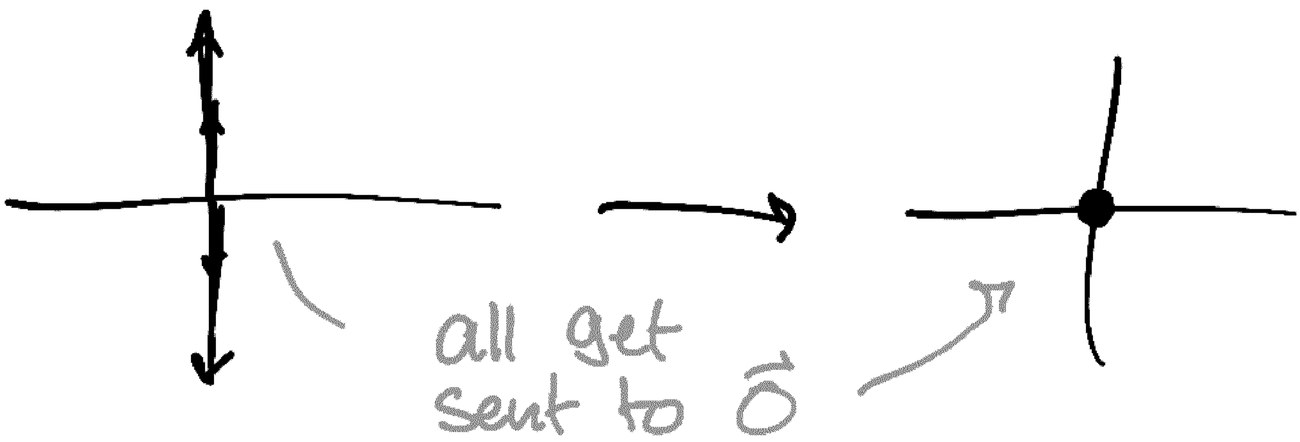


$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}.$$



Which vectors get crushed down to $\vec{0}$? Those along the y-axis!



$$\ker(T) = \operatorname{Span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

<u>Ex</u> $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ can also find the kernel algebraically by solving $A\vec{x} = \vec{0}$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{cases} x_1 = 0 \\ 0 = 0 \end{cases}$$

All solutions are $\begin{bmatrix} 0 \\ x_2 \end{bmatrix}$ where x_2 is an arbitrary scalar.

$\underline{Ex}$: $T: \mathbb{R}^3 \to \mathbb{R}^2$, $T(\vec{x}) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \vec{x}$

Let's find the kernel.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} \underbrace{\begin{matrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{matrix}}_{\text{rref}} & \begin{matrix} 0 \\ 0 \end{matrix} \end{array}\right]$$

$$\begin{cases} x_1 - x_3 = 0 \\ x_2 + 2x_3 = 0 \end{cases}$$

Set $x_3 = t$ (any scalar t)

$\Rightarrow x_1 = t, \; x_2 = -2t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} t$$

So

$\ker(A) = \text{Span}\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right)$.

<u>Properties</u>: If $T: \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation $T(\vec{x}) = A\vec{x}$ for an $(n \times m)$-matrix A.

(a) The zero vector $\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ in $\mathbb{R}^m$ is in $\ker(T)$

(b) $\ker(T)$ is closed under addition

(c) $\ker(T)$ is closed under scalar multiplication.

All of these are either true or false for a given $(n \times n)$-matrix A:

1. A is invertible
2. The linear system $A\vec{x} = \vec{b}$ has a unique solution
3. $\text{rref}(A) = I_n$
4. $\text{rank}(A) = n$
5. $\text{im}(A) = \mathbb{R}^n$
6. $\ker(A) = \{\vec{0}\}$