MAT 211 L6 Mon Sep 14

Recall: If $T:\mathbb{R}^m \to \mathbb{R}^p$, $S:\mathbb{R}^p \to \mathbb{R}^n$ are two linear transformations with

$$T(\vec{x}) = A\vec{x} \text{ and } S(\vec{y}) = B\vec{y} \quad \text{where}$$

A is a $(p\times m)$-matrix, and B is a $(n\times p)$-matrix, then the <u>composition</u> of T and S is a linear transformation

$$S\circ T:\mathbb{R}^m \to \mathbb{R}^n \text{ with}$$

$$(S\circ T)(\vec{x}) = (BA)\vec{x}$$

matrix multiplication

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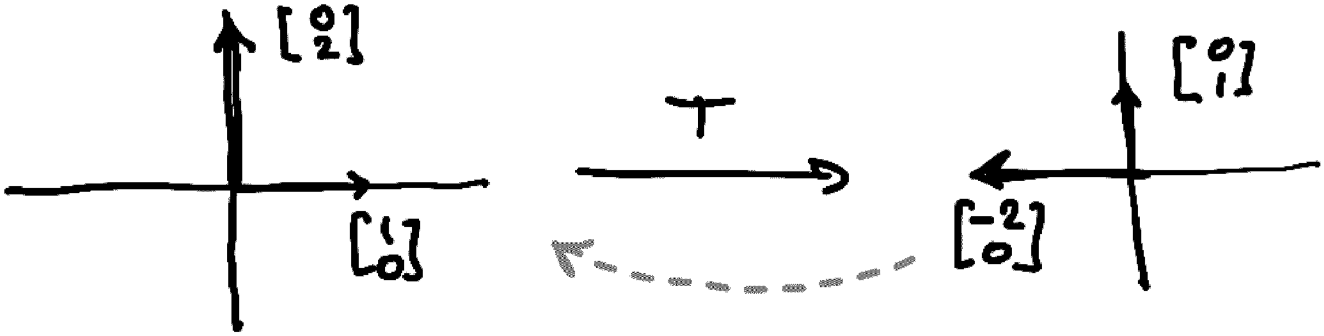
§2.4 <u>Inverse of a linear transformation</u>

<u>Ex</u>: Consider $T:\mathbb{R}^2 \to \mathbb{R}^2$ with

$$T(\vec{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\vec{x}$$



rotation 90° counterclockwise.

How do we undo T? We rotate 90° clockwise.

A rotation 90° CW should send $\begin{bmatrix}1\\0\end{bmatrix}$ to $\begin{bmatrix}0\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ to $\begin{bmatrix}-1\\0\end{bmatrix}$.

$$S\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}0\\-1\end{bmatrix},\quad S\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}1\\0\end{bmatrix}$$

so its matrix is $\begin{bmatrix}0&1\\-1&0\end{bmatrix}$.

$$S(\vec{x})=\begin{bmatrix}0&1\\-1&0\end{bmatrix}\vec{x}.$$

When we now apply T first, followed by S, nothing should happen

$(S\circ T)(\vec{x})=\vec{x}$ for all vectors $\vec{x}$

because first rotating 90° ccw and then rotating 90° cw undos the rotation...

Indeed, we can also find the matrix for $S \circ T$:

$$(S \circ T)(\vec{x}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{= I_2 \text{ (the identity matrix)}} \vec{x} = \vec{x}$$

In this example, if $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ we found a matrix $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ so that $BA = I$.

B is called the <u>inverse to A</u>

(and we will write $B = A^{-1}$)

Recall from Calculus that if f is a function it sometimes has an inverse f^{-1} so that

$$f^{-1}(f(x)) = x, \quad f(f^{-1}(y)) = y.$$

If such f^{-1} exists, f is called <u>invertible</u>.

<u>Def</u>: An $(n \times n)$-matrix A is <u>invertible</u> if there is a matrix B so that $AB = I_n = BA$.

In this case we say A is <u>invertible</u> and write $B = A^{-1}$.

<u>Condition for invertability</u>:

An $(n \times n)$-matrix A is invertible $\iff \mathrm{rref}(A) = I_n \iff \mathrm{rank}(A) = n$.

Matrices being invertible are also important for systems of equations:

Remember that we can write the system

$$\begin{cases} a_{11}x_1 + \cdots + a_{n1}x_n = b_1 \\ \quad \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n = b_n \end{cases}$$

as the matrix equation

$$\begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$A\vec{x} = \vec{b}.$$

Now if the coefficient matrix A has an inverse A^{-1} we can <u>solve for $\vec{x}$</u>

$$A\vec{x} = \vec{b} \Leftrightarrow A^{-1}A\vec{x} = A^{-1}\vec{b}$$

multiply with A^{-1} from the <u>left</u>

$$\Leftrightarrow I_n\vec{x} = A^{-1}\vec{b} \Leftrightarrow \boxed{\vec{x} = A^{-1}\vec{b}}$$

So if A is invertible, there is a <u>unique solution</u> to the system!

<u>How do we compute</u> A^{-1}?

Given an $(n\times n)$-matrix A, we form the matrix

$[A \mid I_n]$ & perform row operations until we arrive at

$[I_n \mid B]$. Then $B = A^{-1}$.

Ex: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Find A^{-1}. We first form $[A \mid I_2]$:

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

$$\qquad A \qquad I_2 \qquad\qquad\qquad I_2 \qquad B$$

So $B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. Let's confirm that this is the inverse!

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \quad \checkmark$$

So $A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

Ex: $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ Find A^{-1} if it exists!

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array}\right]$$

Now we got a zero row, so $\mathrm{rref}(A) \neq I_n$ and we can never arrive at $[I_n | B]$ so A has no inverse.

Algebraic rule:

If A and B are $(n \times n)$-matrices that are both invertible, then AB and BA are both invertible, and

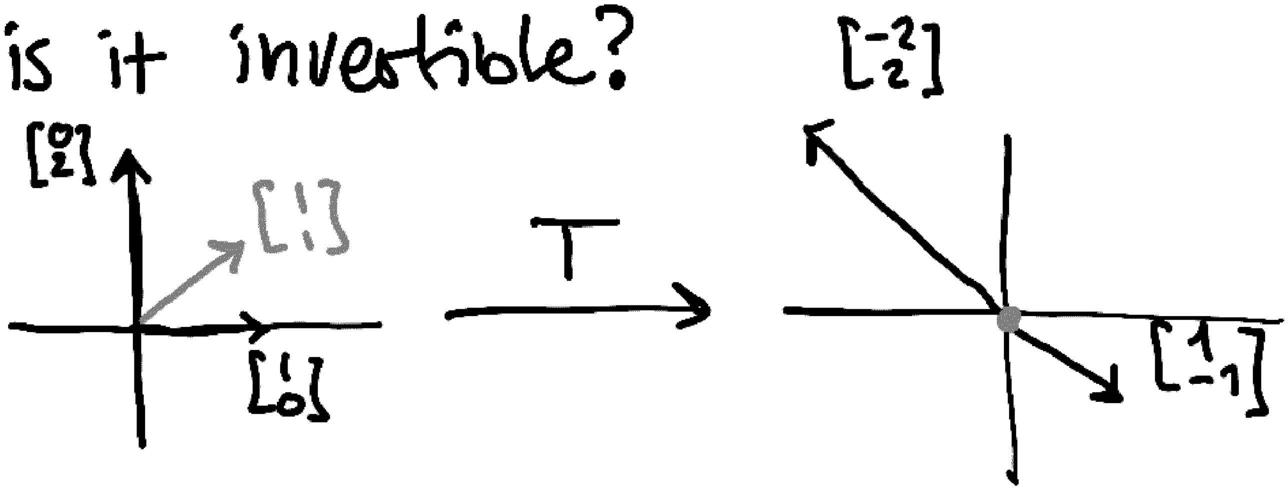
$$(AB)^{-1} = B^{-1}A^{-1}$$
$$(BA)^{-1} = A^{-1}B^{-1}$$

Note how the order of A,B changes!

Ex: Consider the linear transformation

$$T: \mathbb{R}^2 \to \mathbb{R}^2, \quad T(\vec{x}) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \vec{x}$$

What does it do geometrically & is it invertible?



This is an orthogonal projection to the line $y = -x$.

It is <u>not</u> invertible, because both vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ for example gets sent to $\vec{0}$

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = T\left(\begin{bmatrix} -1 \\ -1 \end{bmatrix}\right)$$

and starting with $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, there is no way of undoing this!

We can also come to the same conclusion by showing that the matrix $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ is not invertible.

$$\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{+1} \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

zero row

$\rightsquigarrow \operatorname{rref}\left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right) \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

So we can't continue & A^{-1} does not exist.