MAT 211 LS Wed Sep 9

Recall: • A linear transformation is a function $T: \mathbb{R}^m \to \mathbb{R}^n$ such that $T(\vec{x}) = A\vec{x}$ for some $(n \times m)$-matrix

Ex: Consider linear transformations $T: \mathbb{R}^2 \to \mathbb{R}^2$ and $S: \mathbb{R}^2 \to \mathbb{R}^2$

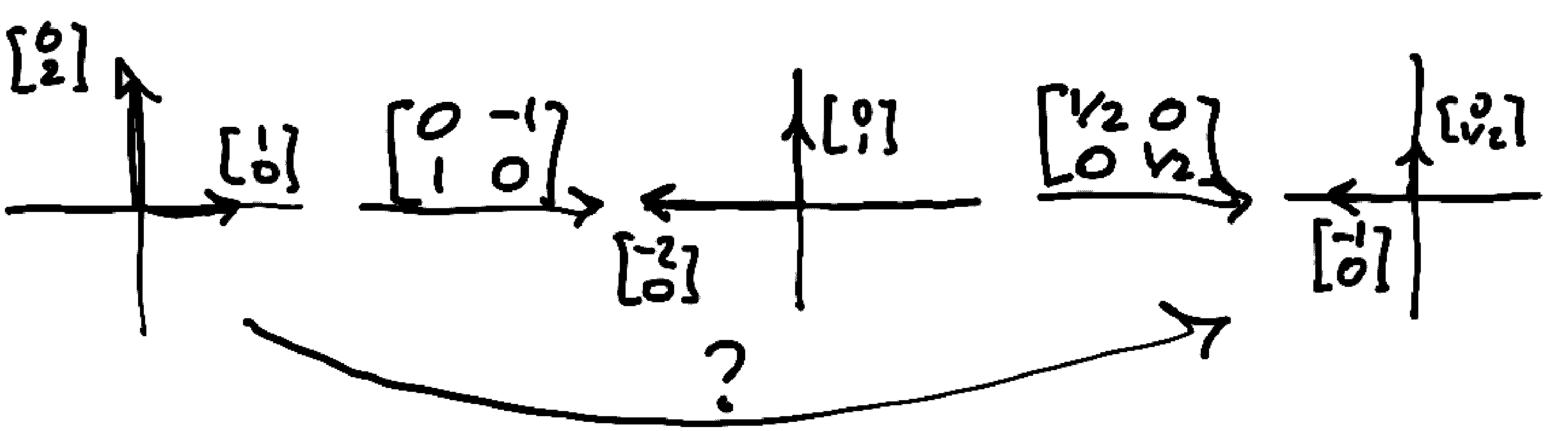
$$T(\vec{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x} \qquad S(\vec{x}) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \vec{x}$$

We compute

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 0 \end{bmatrix},$$

$$S\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}, \quad S\left(\begin{bmatrix} -2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Geometrically:



For the vector $\begin{bmatrix}1\\0\end{bmatrix}$ we first computed $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}0&-1\\1&0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$ and then

$$S\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}1/2&0\\0&1/2\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1/2&0\\0&1/2\end{bmatrix}\begin{bmatrix}0&-1\\1&0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$$
$$=\begin{bmatrix}0\\1/2\end{bmatrix}$$

Can we combine the two matrices $\begin{bmatrix}1/2&0\\0&1/2\end{bmatrix}$ and $\begin{bmatrix}0&-1\\1&0\end{bmatrix}$ to get a new matrix C so that

$$C\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\1/2\end{bmatrix}?$$

Also $C\begin{bmatrix}0\\2\end{bmatrix}=\begin{bmatrix}-1\\0\end{bmatrix}$?

The answer is yes, $C=\begin{bmatrix}0&-1/2\\1/2&0\end{bmatrix}$.

The matrices $\begin{bmatrix}1/2&0\\0&1/2\end{bmatrix}$ and $\begin{bmatrix}0&-1\\1&0\end{bmatrix}$ can be <u>multiplied</u> to get $\begin{bmatrix}0&1/2\\1/2&0\end{bmatrix}$.

<u>Def:</u> Let A be an $(n\times p)$-matrix and let B be a $(q\times m)$-matrix

then their multiplication AB is defined if and only if $p = q$.

\# columns of A = \# rows of B

$$\underset{\substack{n \times p \quad q \times m}}{AB}$$

need to be equal

result is $n \times m$

If A has n rows and $B = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$ has n columns, then

$$AB = A \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ A\vec{v}_1 & \cdots & A\vec{v}_n \\ | & & | \end{bmatrix}.$$

<u>Ex:</u> $A = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

2×2 ← equal → 2×2

So we can multiply them:

$$AB = A\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} = \begin{bmatrix} | & | \\ A\begin{bmatrix}0\\1\end{bmatrix} & A\begin{bmatrix}-1\\0\end{bmatrix} \\ | & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | \\ \begin{bmatrix}1/2 & 0\\ 0 & 1/2\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} & \begin{bmatrix}1/2 & 0\\ 0 & 1/2\end{bmatrix}\begin{bmatrix}-1\\0\end{bmatrix} \\ | & | \end{bmatrix} = \begin{bmatrix}0 & -1/2\\ 1/2 & 0\end{bmatrix}$$

<u>Ex</u>: $A = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$, $B = \begin{bmatrix}2 & 1\\ 0 & 1\end{bmatrix}$.

$$AB = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}2 & 1\\ 0 & 1\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$$

$$BA = \begin{bmatrix}2 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 2\\ 0 & 0\end{bmatrix}$$

Note AB and BA are <u>different!</u>

<u>Theorem</u>: If $T: \mathbb{R}^m \to \mathbb{R}^p$ and $S: \mathbb{R}^p \to \mathbb{R}^n$ are two linear transformations. If $T(\vec{x}) = \underset{p\times m}{A}\vec{x}$ and $S(\vec{x}) = \underset{n\times p}{B}\vec{x}$ then their composition $S\circ T: \mathbb{R}^m \to \mathbb{R}^n$ is given by

$$\boxed{(S \circ T)(\vec{x}) = (BA)\vec{x}}$$

<u>Ex</u>: Consider linear transformations $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ and $S: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$T(\vec{x}) = \underbrace{\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}}_{A} \vec{x} \qquad S(\vec{y}) = \underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 4 \end{bmatrix}}_{B} \vec{y}$$

Their composition is

$\mathbb{R}^2 \xrightarrow{T} \mathbb{R}^3 \xrightarrow{S} \mathbb{R}^2$ and the matrix is

$$BA = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} B\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} & B\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ 10 & 1 \end{bmatrix}$$

<u>Matrix algebra</u>:

Recall that the identity matrix

is $I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

$\left(\text{e.g. } I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \dots \right)$

Then if A is an $(n \times m)$-matrix, then

$$\boxed{A I_m = I_n A = A}$$

We have:

$$\boxed{A(BC) = (AB)C}$$

assuming all products are defined (i.e. the matrices have the correct sizes.)

If A, B $n \times p$ and C, D are $p \times m$

Then

$$\boxed{\begin{aligned} A(C+D) &= AC+AD \\ (A+B)C &= AC+BC \end{aligned}}$$

If k is a scalar, A is $n\times p$ and B is $p\times m$, then

$$\boxed{(kA)B = A(kB) = k(AB).}$$

As we observed in the example above, if A and B are $n\times n$ then <u>generally</u>

$$\boxed{AB \neq BA}$$

"matrix multiplication is <u>not</u> commutative"

<u>Ex</u>: Do $A=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$ and $B=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}$ commute?

$$AB=\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}2&0\\0&-1\end{bmatrix}=\left[\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}2\\0\end{bmatrix}\quad\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}0\\-1\end{bmatrix}\right]$$

$$=\begin{bmatrix}2&0\\0&-2\end{bmatrix}$$

$$BA=\begin{bmatrix}2&0\\0&-1\end{bmatrix}\begin{bmatrix}1&0\\0&2\end{bmatrix}=\begin{bmatrix}2&0\\0&-2\end{bmatrix}$$

Answer: Yes!

What about A and $C=\begin{bmatrix}0&0\\1&0\end{bmatrix}$?

$$AC=\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}0&0\\1&0\end{bmatrix}=\begin{bmatrix}0&0\\2&0\end{bmatrix}$$

$$CA=\begin{bmatrix}0&0\\1&0\end{bmatrix}\begin{bmatrix}1&0\\0&2\end{bmatrix}=\begin{bmatrix}0&0\\1&0\end{bmatrix}$$

Can we find <u>all</u> matrices D that commutes with A?

Set $D=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ and let's look at the equation

$$AD=DA$$

$$\Leftrightarrow \begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}=\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}1&0\\0&2\end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} a & b \\ 2c & 2d \end{bmatrix} = \begin{bmatrix} a & 2b \\ c & 2d \end{bmatrix}$$

$$\begin{cases} a = a \\ b = 2b \\ 2c = c \\ 2d = 2d \end{cases} \Rightarrow \quad b = c = 0$$

and <u>any</u> a, d are allowed.

So all matrices that commute with $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ are those of the form $D = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ for arbitrary real numbers a and d.
