Recall: • A linear transformation is a function $T: \mathbb{R}^m \to \mathbb{R}^n$ such that $T(\vec{x}) = A\vec{x}$ for some $(n \times m)$-matrix

• The matrix A for a linear transformation satisfies

$$A = \begin{bmatrix} | & & | \\ T(\vec{e}_1) & \cdots & T(\vec{e}_m) \\ | & & | \end{bmatrix} \qquad \vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{i-th component}$$

• A linear transformation T satisfies

$T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$

$T(k\vec{v}) = kT(\vec{v})$ for any scalar k

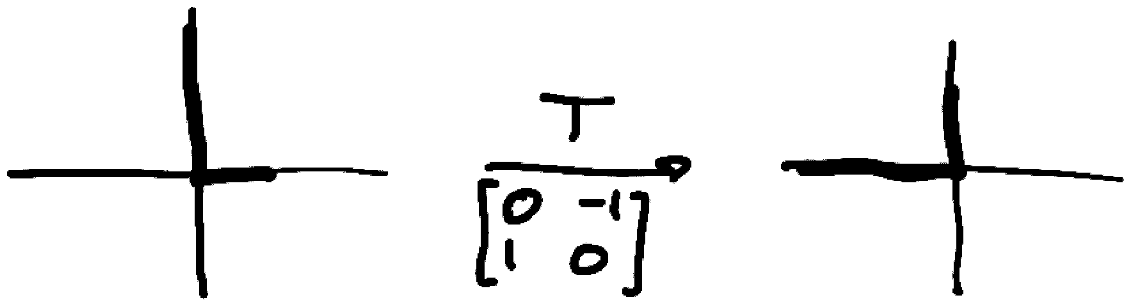
§2.2 Linear transformations in geometry

We would now like to get a better geometric understanding of linear transformations

<u>Ex</u>: Last time we already saw

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2 : T(\vec{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x}$

and how it is a counterclockwise rotation in the plane by $90°$.

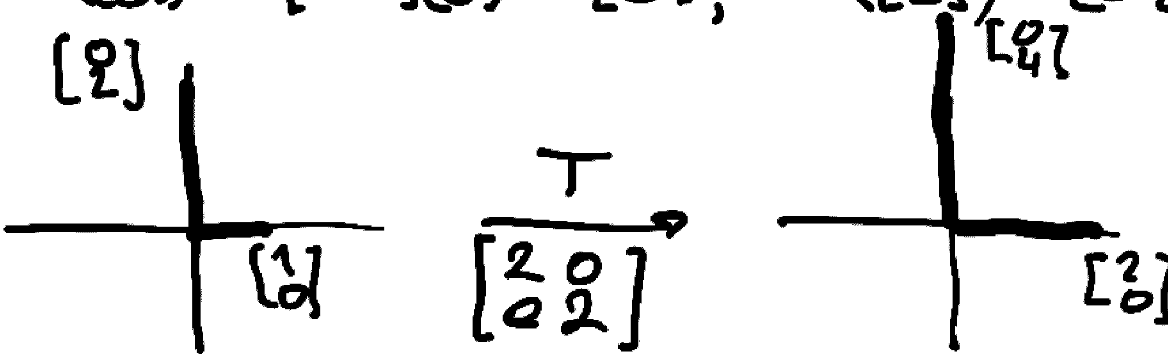


<u>Ex</u>: Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and let's figure out how these matrices act on vectors, and the "L" as before.

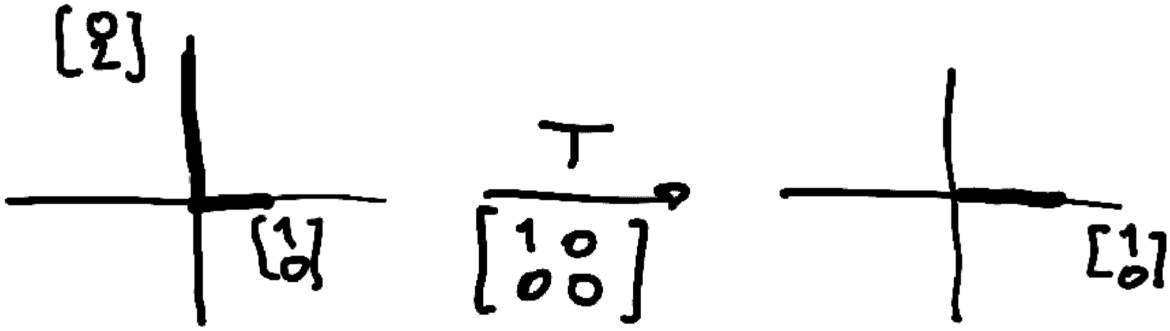
1. $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $T(\vec{x}) = A\vec{x} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \vec{x}$

$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$



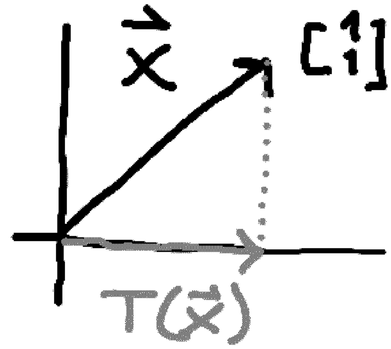
② $T(\vec{x}) = B\vec{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\vec{x}$.

$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$



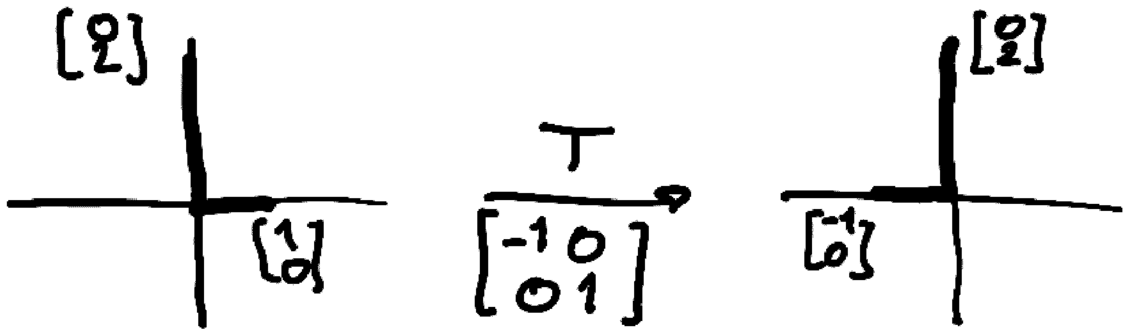
It crushes the vector $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
Generally it projects all vectors to the x-axis:

$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$



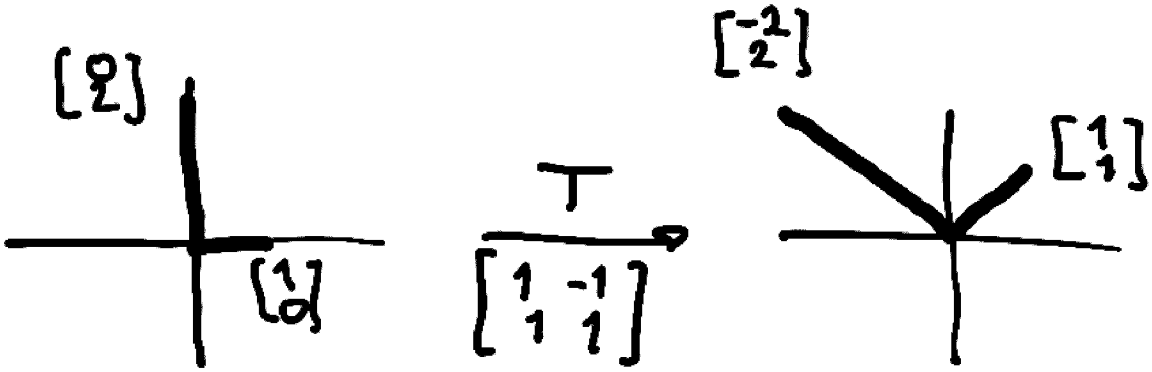
③ $T(\vec{x}) = C\vec{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\vec{x}$

$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$



(4) $T(\vec{x}) = D\vec{x} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \vec{x}$.

$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$



Rotation counterclockwise $45°$.

<u>Scalings</u>: For any positive scalar k the linear transformation corresponding to the matrix $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ will scale every vector by k:

$$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} kx_1 \\ kx_2 \end{bmatrix} = k \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

If $k > 1$ it will <u>increase</u> the length ab vectors by a factor k.

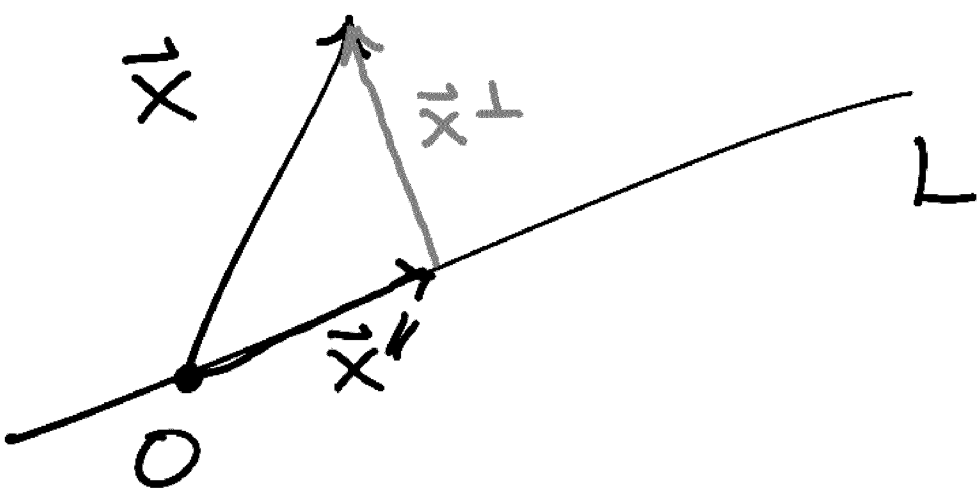
If $0 < k < 1$, it will <u>decrease</u> the length of vectors by a factor k.

Orthogonal projections

If L is a line in the plane, we can always write any vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in the plane as

$$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$$

Where $\left| \begin{array}{l} \vec{x}^{\parallel} = \text{vector parallel to } L \\ \vec{x}^{\perp} = \text{vector perpendicular to } L \end{array} \right.$



We define the orthogonal projection onto the line L to be

$$\text{proj}_L(\vec{x}) = \vec{x}^{\parallel}.$$

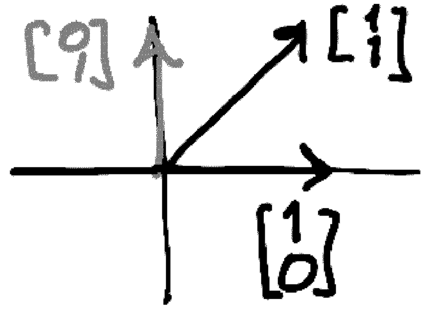
It's the "shadow" of $\vec{x}$ on L.

We will work out a better formula later in the course.

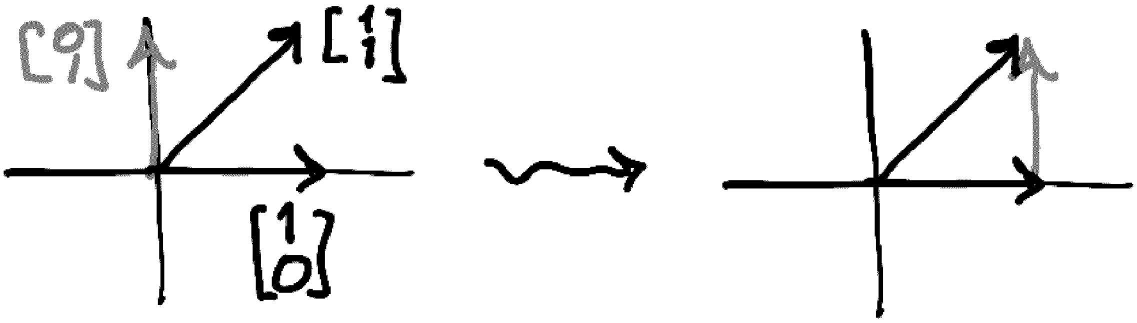
Crash course in geometry of vectors

We have seen how we add vectors componentwise: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \end{bmatrix}$

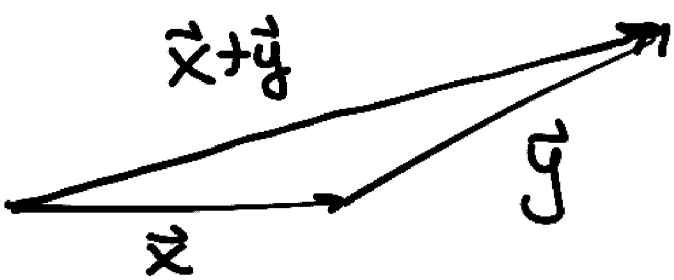
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



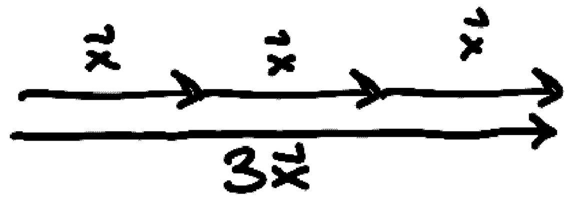
Geometrically we add vectors by placing them head-to-tail:



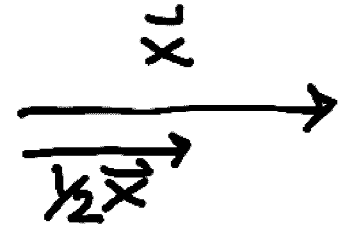
Moving a vector so it does not start at the origin is ok as long as we don't change its direction.



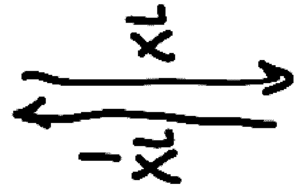
$3\vec{x} = \vec{x} + \vec{x} + \vec{x}$



$\frac{1}{2}\vec{x}$:



$-\vec{x}$ is $\vec{x}$ with the opposite direction:

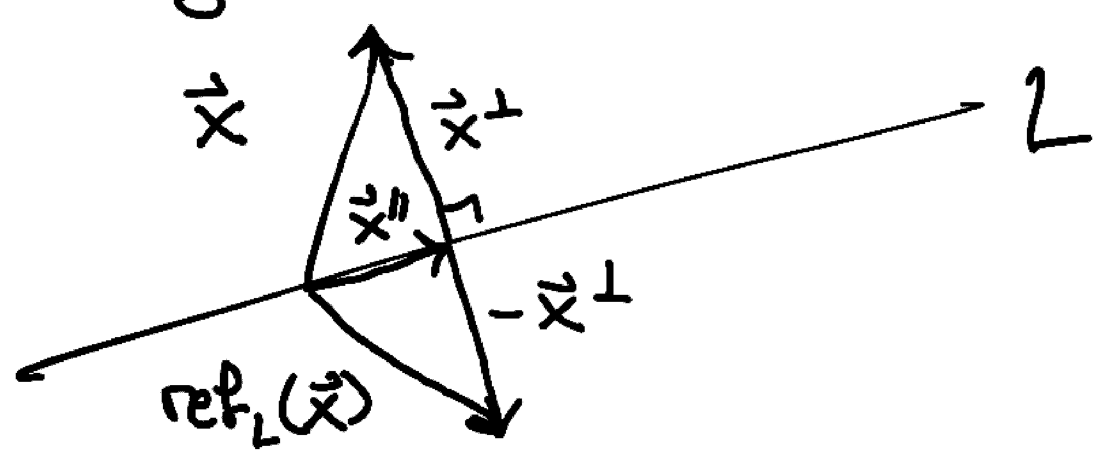


Back to the linear transformations:

<u>Reflections:</u>

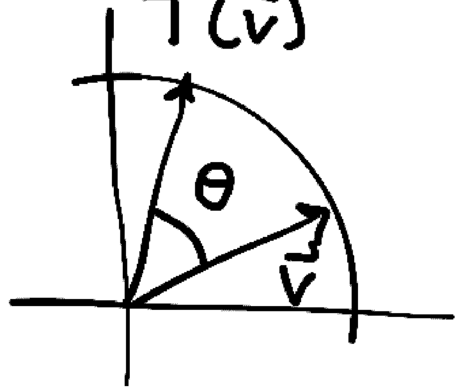
These are kind of similar to orthogonal projections.

If L is again a line in the plane:



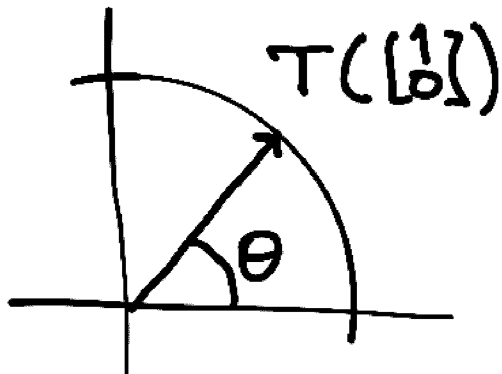
$$\text{ref}_L(\vec{x}) = \underbrace{\vec{x}^{\parallel}}_{\text{proj}_L(\vec{x})} - \vec{x}^{\perp}.$$

Rotations :



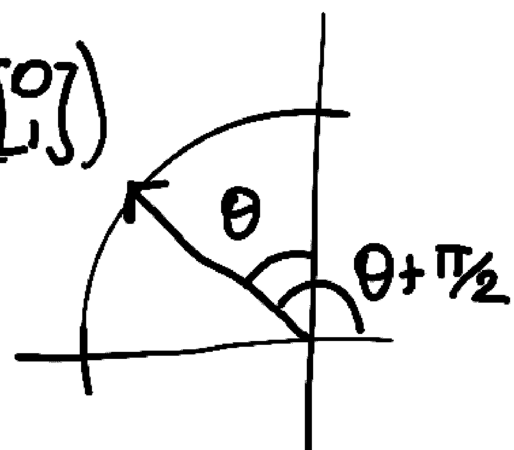
A rotation counterclockwise by angle θ in the plane.

Let $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$:



This vector is $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = T(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$

Similarly if $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $T(\begin{bmatrix} 0 \\ 1 \end{bmatrix})$



This vector is

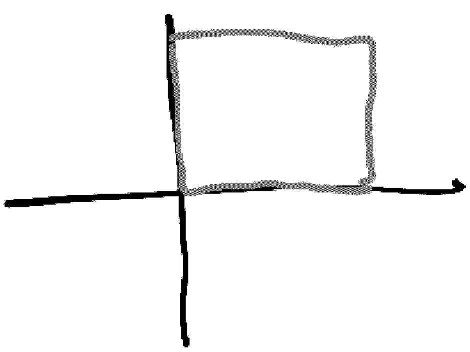
$$\begin{bmatrix} \cos(\theta+\pi/2) \\ \sin(\theta+\pi/2) \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = T(\begin{bmatrix} 0 \\ 1 \end{bmatrix})$$

So the matrix that gives

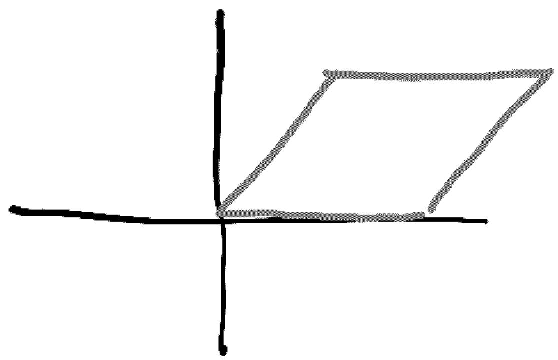
rotation in the plane by θ radians counterclockwise is

$$A = \begin{bmatrix} | & | \\ T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) & T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) \\ | & | \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

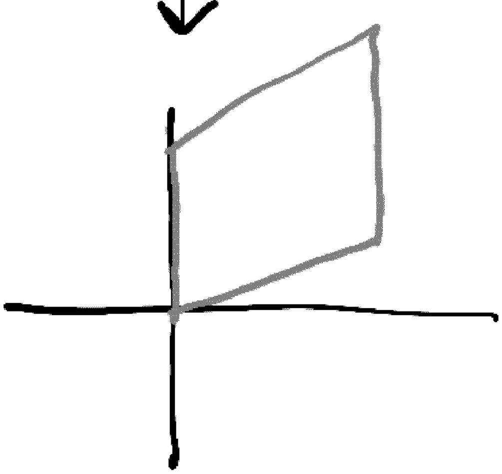
Shears:



$\xrightarrow[\text{shear}]{\text{horizontal}}$



$\Big\downarrow$ vertical shear



If k is a scalar, these are represented by

$$\underset{\text{horizontal}}{\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}} \quad \text{or} \quad \underset{\text{vertical}}{\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}}$$
