<u>Recall</u>:

Systems of equs

$$\begin{cases} a_{11}x_1 + \cdots + a_{1m}x_m = b_1 \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nm}x_m = b_m \end{cases}$$

Can be described by the matrix equation

$\boxed{A\vec{x} = \vec{b}}$, where $A = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}$

$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$, $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$

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§2.1 <u>Linear transformations</u>

<u>Ex</u>: Suppose we have a vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ that records our position in the plane. To make it harder to find us, we will encode our position

to $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ where $\begin{cases} y_1 = x_1 + 3x_2 \\ y_2 = 2x_1 + 5x_2 \end{cases}$ $(*)$

So if $\vec{x} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$, our encoded position is $\vec{y} = \begin{bmatrix} 3+3\cdot 7 \\ 2\cdot 3+5\cdot 7 \end{bmatrix} = \begin{bmatrix} 24 \\ 41 \end{bmatrix}$.

Our encoding process $(*)$ can be described by a matrix equation

$$\boxed{A\vec{x} = \vec{y}}, \quad A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

Example of a <u>linear transformation</u>

<u>Ex</u>: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x$ is a function that multiplies our input by 2.

<u>Ex</u>: $T : \mathbb{R}^2 \to \mathbb{R}^2$ $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$

$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Multiplies our input vector by $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

__Def__: A function T from $\mathbb{R}^m$ to $\mathbb{R}^n$ (written $T: \mathbb{R}^m \longrightarrow \mathbb{R}^n$) is called a __linear transformation__ if there exists an $(n \times m)$-matrix A such that $\boxed{T(\vec{x}) = A\vec{x}}$

__Note__: Input $\vec{x}$ is a vector in $\mathbb{R}^m$ (so m components), $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$

A has size $n \times m$

$\underset{n\times m}{A}\ \underset{m\times 1}{\vec{x}}$ — result is $n \times 1$, so a vector in $\mathbb{R}^n$.

__Ex__: $A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & 2 & 5 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$A\vec{x} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 + 2x_3 \\ 3x_1 + 2x_2 + 5x_3 \end{bmatrix} = T(\vec{x})$$

$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

Ex: $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $T(\vec{x}) = \vec{x}$ is also in fact a linear transformation.

This is because

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called the identity matrix.

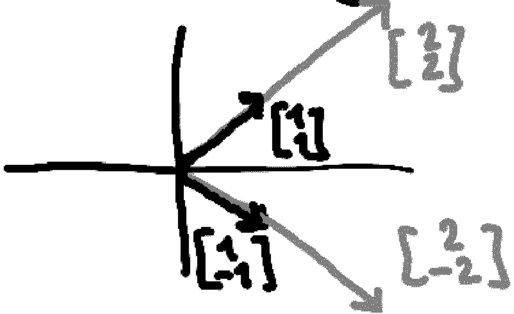
$$\underset{3\times 3}{I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}, \ldots, \underset{n \times n}{I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}}.$$

Ex: Let's consider the linear transformation

$$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad T(\vec{x}) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

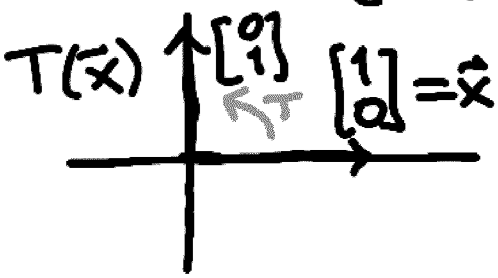
$= \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$. E.g. $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

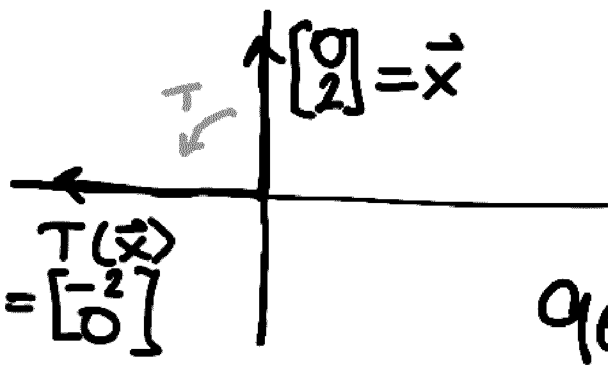


Ex: $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $T(\vec{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x}$.

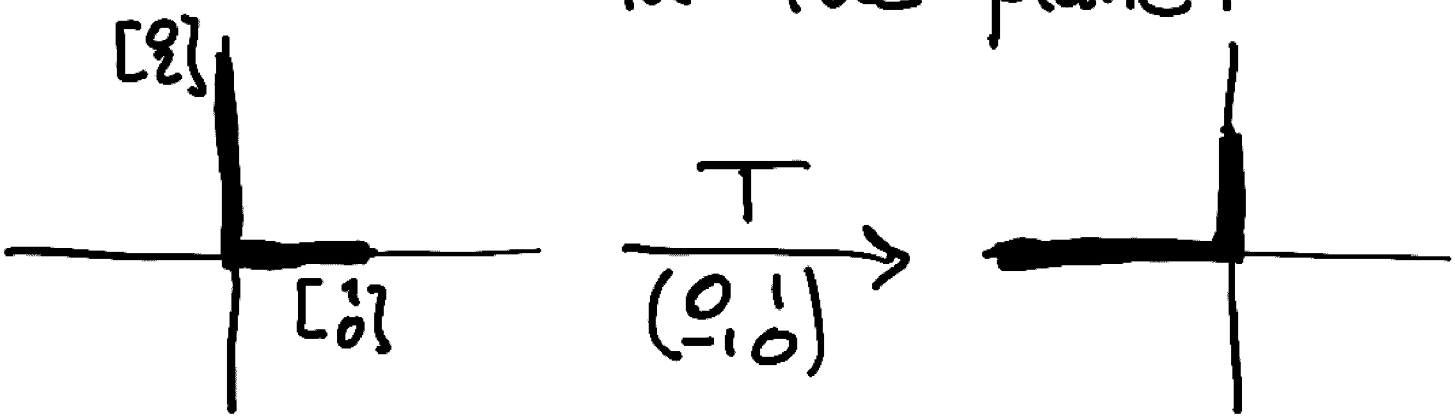
Ex: $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



$$T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$



T seems to rotate vectors by 90° counterclockwise in the plane!



Ex: Now let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

& consider the linear transformation

$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $T(\vec{x}) = A\vec{x}$.

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\4\\7\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}2\\5\\8\end{bmatrix}$$

This phenomenon is actually general!

<u>Theorem</u>: If $T: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is a linear transformation $T(\vec{x}) = A\vec{x}$, then

$$A = \begin{bmatrix} | & & | \\ T(\vec{e}_1) & \cdots & T(\vec{e}_n) \\ | & & | \end{bmatrix}, \quad \vec{e}_i = \begin{bmatrix}0\\ \vdots\\0\\1\\0\\ \vdots\\0\end{bmatrix}$$

i-th position

The vectors $\vec{e}_1 = \begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix}0\\1\\0\\ \vdots\\0\end{bmatrix}$, $\ldots$, $\vec{e}_n = \begin{bmatrix}0\\ \vdots\\0\\1\end{bmatrix}$ are sometimes called the <u>standard vectors</u> in $\mathbb{R}^n$.

Ex: If $T: \mathbb{R}^m \to \mathbb{R}^n$, what's the relation between $T(\vec{v})$, $T(\vec{w})$, and $T(\vec{v}+\vec{w})$?

We know that T is linear, so $T(\vec{x}) = A\vec{x}$ for any vector $\vec{x}$ in $\mathbb{R}^m$.

$$T(\vec{v}) = A\vec{v}, \quad T(\vec{w}) = A\vec{w}$$

$$T(\vec{v}+\vec{w}) = A(\vec{v}+\vec{w}) = A\vec{v} + A\vec{w}$$
$$= T(\vec{v}) + T(\vec{w}).$$

so $\boxed{T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})}$ 1

Similarly, what about $T(k\vec{v})$ where k is a real number?

$$T(k\vec{v}) = A(k\vec{v}) = kA\vec{v} = kT(\vec{v}).$$

so $\boxed{T(k\vec{v}) = kT(\vec{v})}$ 2

It's acually true that a transformation $T: \mathbb{R}^m \to \mathbb{R}^n$ is linear if and only if ① & ② are true!

<u>Ex</u>: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\2\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}5\\2\end{bmatrix}$.

① Find $T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right)$ and $T\left(\begin{bmatrix}2\\1\end{bmatrix}\right)$

② Find the matrix A such that $T(\vec{x}) = A\vec{x}$.

<u>Sol</u>: ① $\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}$ so

$$T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) - T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\2\end{bmatrix} - \begin{bmatrix}5\\2\end{bmatrix} = \begin{bmatrix}-5\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = T\left(2\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\right) = 2T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$
$$= 2\begin{bmatrix}0\\2\end{bmatrix} + \begin{bmatrix}5\\2\end{bmatrix} = \begin{bmatrix}5\\6\end{bmatrix}.$$

② We know $A = \begin{bmatrix} | & | \\ T(\vec{e}_1) & T(\vec{e}_2) \\ | & | \end{bmatrix}$,

$\vec{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$, and so $A = \begin{bmatrix}0 & 5\\2 & 2\end{bmatrix}$.
