Recall: If A is an $n \times n$-matrix, the eigenvalues of A are the solutions to the characteristic equation:

$$\det(A - \lambda I_n) = 0$$

The algebraic multiplicity of an eigenvalue λ is its multiplicity of λ as a root of $\det(A - \lambda I)$.

Ex: $A = \begin{bmatrix} -3 & 0 & 4 \\ 0 & -1 & 0 \\ -2 & 7 & 3 \end{bmatrix}$. Let's find the eigenvalues for A, with algebraic multiplicities.

$$\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & 0 & 4 \\ 0 & -1-\lambda & 0 \\ -2 & 7 & 3-\lambda \end{vmatrix}$$

$$= (-1-\lambda) \begin{vmatrix} -3-\lambda & 4 \\ -2 & 3-\lambda \end{vmatrix} = (-1-\lambda)[(-3-\lambda)(3-\lambda) + 8]$$

$$= -(1+\lambda)\left[-(9-\lambda^2)+8\right] = -(1+\lambda)(\lambda^2-1)$$

$$= -(1+\lambda)^2(\lambda-1) = 0$$

Eigenvalue | $\lambda = 1$, alg. mult $= 1$
$\lambda = -1$, alg. mult $= 2$

The determinant $\det(A-\lambda I)$ is a polynomial of degree n, so counting with multiplicities, there are at most n roots.

If n is odd, the polynomial always have at least 1 root.

Thm: An $(n \times n)$-matrix A has at most n eigenvalues, counted with algebraic multiplicities.

If n is odd, there is at least one eigenvalue.

If A has n eigenvalues, we are able to compute the determinant & trace of A!

Thm: If A is an $(n \times n)$-matrix, that has n eigenvalues $\lambda_1, \dots, \lambda_n$, then ← these may have repeats!

$$\det A = \lambda_1 \cdot \dots \cdot \lambda_n$$

$$\operatorname{tr} A = \lambda_1 + \dots + \lambda_n$$

Remark: These formulas are false if A has fewer than n eigenvalues (counted w/ alg. mult.).

Ex:

$$A = \begin{bmatrix} -3 & 0 & 6 \\ 0 & -1 & 0 \\ -2 & 7 & 3 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & 0 & 6 \\ 0 & -1-\lambda & 0 \\ -2 & 7 & 3-\lambda \end{vmatrix} \leftarrow$$

$$= (-1-\lambda) \begin{vmatrix} -3-\lambda & 6 \\ -2 & 3-\lambda \end{vmatrix}$$

$$= (-1-\lambda)\left[(-3-\lambda)(3-\lambda)+12\right]$$

$$= (-1-\lambda)\left[-(9-\lambda^2)+12\right] = -(1+\lambda)(\lambda^2+3)$$

$$= 0$$

So $n=3$ but there's only 1 eigenvalue $\lambda=-1$ with alg. mult. 1

$$\det A = \begin{vmatrix} -3 & 0 & 6 \\ 0 & -1 & 0 \\ -2 & 7 & 3 \end{vmatrix}$$

$$= -\begin{vmatrix} -3 & 6 \\ -2 & 3 \end{vmatrix} = -(-9+12) = -3$$

So $\det A$ is not product of eigenvalues.

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§7.3 Finding eigenvectors

Now that we know how to find eigenvalues, how do we find eigenvectors?

If λ is an eigenvalue, and $\vec{v} \neq \vec{0}$ is an eigenvector we have

$$A\vec{v} = \lambda\vec{v} \Leftrightarrow (A - \lambda I)\vec{v} = \vec{0}.$$

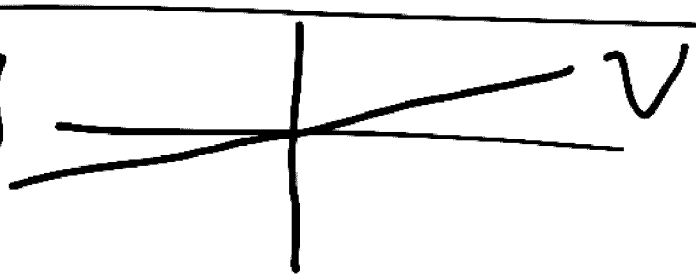
so $\vec{v}$ belongs to $\ker(A - \lambda I)$.

So to find $\vec{v}$, we need to find $\boxed{\ker(A - \lambda I)}$.

Def: Consider an eigenvalue λ of an $(n \times n)$-matrix. The <u>eigenspace associated with λ</u> is

$$\boxed{E_\lambda = \ker(A - \lambda I)}$$

Ex: $V = \text{span}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$



$A = \frac{1}{10}\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$,

last lecture we computed the

eigenvalues & they are $\lambda=1$ and $\lambda=0$.

Now lets find the eigenspaces E_1 and E_0.

$$E_1 = \ker(A-I) = \ker\begin{bmatrix} \frac{9}{10}-1 & \frac{3}{10} \\ \frac{3}{10} & \frac{1}{10}-1 \end{bmatrix}$$

$$= \ker\left(\frac{1}{10}\begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix}\right)$$

TO find ker, we first compute rref.

$$\frac{1}{10}\begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} \xrightarrow{\cdot 3} \frac{1}{10}\begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix} \xrightarrow{\cdot(-1)}$$

$$\longrightarrow \frac{1}{10}\begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} = \text{rref}$$

$\text{rref} \cdot \vec{x} = \vec{0}$ then is the eqn $\frac{1}{10}(x_1 - 3x_2) = 0$.

Setting $x_2 = t$ gives $x_1 = 3t$

So $\vec{x} = \begin{bmatrix} 3t \\ t \end{bmatrix} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$E_1 = \left\{ t\begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\} = \text{Span}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = V$

$E_0 = \ker(A - 0 \cdot I) = \ker A$

$A = \frac{1}{10}\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \cdot \frac{1}{3} \longrightarrow \frac{1}{10}\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}^{-1}$

$\longrightarrow \frac{1}{10}\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} = \text{rref}(A)$

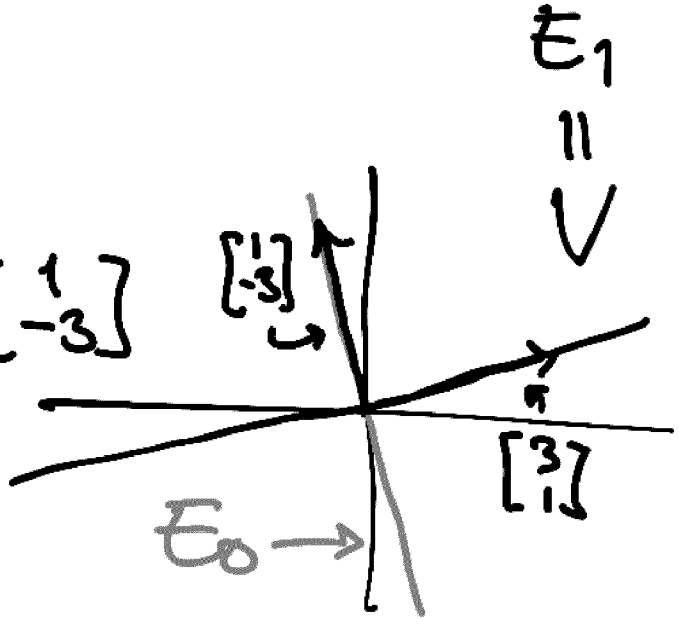
$\text{rref}(A)\vec{x} = \vec{0}$ gives

$3x_1 + x_2 = 0$.

$x_1 = t$ gives $x_2 = -3t$

$\vec{x} = \begin{bmatrix} t \\ -3t \end{bmatrix} = t\begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$E_0 = \left\{ t\begin{bmatrix} 1 \\ -3 \end{bmatrix} \right\} = \text{Span}\begin{bmatrix} 1 \\ -3 \end{bmatrix}$



So any non-zero vector in E_0 is an eigenvector for proj_V w/ eigenvalue $\lambda=0$, eg., $\begin{bmatrix}1\\-3\end{bmatrix}$.
Similarly any non-zero vector in E_1 is an eigenvector for proj_V w/ eigenvalue $\lambda=1$, e.g., $\begin{bmatrix}3\\1\end{bmatrix}$.

Recall relation w/ diagonalization of matrices: If we can find a basis $B=(\vec{v}_1,\dots,\vec{v}_n)$ for $\mathbb{R}^n$ of eigenvectors for A, then set $S=\begin{bmatrix}\uparrow & & \uparrow\\ \vec{v}_1 & \cdots & \vec{v}_n\\ \downarrow & & \downarrow\end{bmatrix}$ &

$[A]_B = S^{-1}AS$ will be diagonal

$$=\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}$$

λ_i = eigenvalue for $\vec{v}_i$.

EX: Find the eigenspaces of $A=\begin{bmatrix}1&1&1\\0&0&1\\0&0&1\end{bmatrix}$. Is A diagonalizable?

Sol: We first find eigenvalues:

$$\det(A-\lambda I)=\begin{vmatrix}1-\lambda&1&1\\0&-\lambda&1\\0&0&1-\lambda\end{vmatrix}$$

$$=(1-\lambda)(-\lambda)(1-\lambda)=-\lambda(1-\lambda)^2=0$$

$\lambda_1=0$, $\lambda_{2,3}=1$ (eigenvalue 1 has multiplicity 2)

$E_0=\ker A$.

$$A=\begin{bmatrix}1&1&1\\0&0&1\\0&0&1\end{bmatrix}\begin{matrix}\leftarrow\\ \\ \leftarrow\end{matrix}\begin{matrix}\\ \\ -1\end{matrix}\begin{matrix}\\ \\ -1\end{matrix}\longrightarrow\begin{bmatrix}1&1&0\\0&0&1\\0&0&0\end{bmatrix}=\text{rref}(A)$$

We now solve $\text{rref}(A)\vec{x}=\vec{0}$

$$\Leftrightarrow\begin{cases}x_1+x_2=0\\ x_3=0\end{cases}\quad \text{Set } x_2=t.$$

$\Rightarrow x_1 = -t$, so $\vec{x} = \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$E_0 = \text{span} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

$E_1 = \ker(A - I) = \ker \underbrace{\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{call this } A_1}$

$A_1 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ \end{matrix} \begin{matrix} \\ \\ \end{matrix} \begin{matrix} \\ \\ +1 \end{matrix} \longrightarrow \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ -\frac{1}{2} \end{matrix}$

$\longrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{rref } A_1$

Solve $(\text{rref } A_1)\vec{x} = \vec{0}$.

$\begin{cases} x_2 = 0 \\ x_3 = 0 \end{cases}$ x_1 is arbitrary!

$x_1 = t$

$\vec{x} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$E_1 = \ker A_1 = \text{span} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$E_0 = \text{span}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Note $\dim E_0 = 1$

$\dim E_1 = 1$

If A were diagonalizable, we would be able to find 3 linearly independent eigenvectors we can pick $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

but no matter how we pick a 3rd eigenvector, it will not be linearly independent from $\vec{v}_1, \vec{v}_2$.

Ans: A is <u>not</u> diagonalizable, Since we <u>cannot</u> find a basis for $\mathbb{R}^3$ of eigenvectors for A.