Recall: • $T: \mathbb{R}^n \to \mathbb{R}^n$, $T(\vec{x}) = A\vec{x}$ is diagonalizable if we can find an invertible matrix S such that $S^{-1}AS$ is diagonal.

Equivalently: T diagonalizable if there's a basis B for $\mathbb{R}^n$ such that $[A]_B$ is diagonal.

• A vector $\vec{v} \neq \vec{0}$ is called an eigenvector for T if $T(\vec{v}) = \lambda\vec{v}$ for some scalar λ. λ is called the eigenvalue for $\vec{v}$.

• T is diagonalizable $\iff$ there exists a basis $B = (\vec{v}_1, \ldots, \vec{v}_n)$ of $\mathbb{R}^n$ where each $\vec{v}_i$ is an eigenvector for T.

Ex: If A is an orthogonal matrix, and $\vec{v}$ is an eigenvector for A, what are the possible eigenvalues for $\vec{v}$?

Sol: $\vec{v}$ is an eigenvector means

(*) $A\vec{v} = \lambda\vec{v}$. That A is orthogonal means $\|A\vec{v}\| = \|\vec{v}\|$. Taking lengths in (*) gives

$$\|A\vec{v}\| = \|\lambda\vec{v}\| \Leftrightarrow \|\vec{v}\| = |\lambda| \cdot \|\vec{v}\|$$

$$\Leftrightarrow 1 = |\lambda| \quad (\text{since } \vec{v} \neq \vec{0})$$

$$\Leftrightarrow \boxed{\lambda = \pm 1}$$

Answer is that the only possible eigenvalues are $\lambda = \pm 1$.

Ex: Which matrices A has 0 as an eigenvalue?

Sol: If 0 is an eigenvalue, then it means $A\vec{v}=0\cdot\vec{v}=\vec{0}$ for some $\vec{v}\neq\vec{0}$. So $\vec{v}$ is in $\ker A$, and $\ker A \neq \{0\}$, which means A is not invertible.

Actually the opposite is also true: If A is invertible, $\ker A \neq \{0\}$. So there exists some $\vec{v}\neq\vec{0}$ in $\ker A$, or equivalently $A\vec{v}=\vec{0}$.

This gives new characterizations on when a matrix is invertible. Let us summarize it:

Theorem: For an $(n\times n)$-matrix, the following are equivalent:

(1) A is invertible

(2) The equation $A\vec{x}=\vec{b}$ has a unique

Solution $\vec{x}$ for all $\vec{b}$ in $\mathbb{R}^n$

(3) $\operatorname{rref} A = I_n$

(4) $\operatorname{rank} A = n$

(5) $\operatorname{im} A = \mathbb{R}^n$

(6) $\ker A = \{0\}$

(7) The columns of A form a basis of $\mathbb{R}^n$

(8) The columns of A span $\mathbb{R}^n$

(9) The columns of A are linearly independent

(10) $\det A \neq 0$

(11) 0 is not an eigenvalue of A.

We saw that A diagonalizable $\Leftrightarrow$ there exists an eigenbasis.

Our first task is to learn how to find eigenvalues/eigenvectors

§7.2 Finding eigenvalues

Let A be an $(n \times n)$-matrix.

λ is an eigenvalue for A

$\Leftrightarrow$ $A\vec{v}=\lambda\vec{v}$ for some $\vec{v}\neq\vec{0}$.

$\Leftrightarrow$ $A\vec{v}-\lambda\vec{v}=\vec{0}$

$\Leftrightarrow$ $\underbrace{(A-\lambda I_n)}_{n\times n\text{-matrix}}\vec{v}=\vec{0}$

$\Leftrightarrow$ $\ker(A-\lambda I)\neq\{0\}$

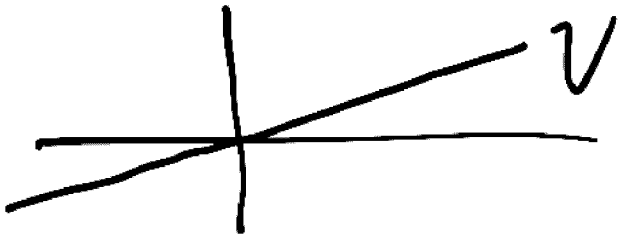
(namely, it contains $\vec{v}\neq\vec{0}$!)

$\Leftrightarrow$ $A-\lambda I$ is <u>not</u> invertible

$\Leftrightarrow$ $\det(A-\lambda I)=0$.

<u>Theorem</u>: A scalar λ is an eigenvalue for an $(n\times n)$-matrix A if and only if $\det(A-\lambda I)=0$.
The equation $\det(A-\lambda I)=0$ is called the <u>characteristic equation</u> of A.

Ex: $V = \text{span}\begin{bmatrix}3\\1\end{bmatrix}$,



$\text{Proj}_V : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

Saw last time that its matrix is

$$A = \frac{1}{10}\begin{bmatrix}9 & 3\\ 3 & 1\end{bmatrix}.$$

Using the characteristic equ we can find the eigenvalues:

$$\det(A-\lambda I) = \det\left(\begin{bmatrix}9/10 & 3/10\\ 3/10 & 1/10\end{bmatrix} - \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right)$$

$$= \begin{vmatrix}\frac{9}{10}-\lambda & \frac{3}{10}\\ \frac{3}{10} & \frac{1}{10}-\lambda\end{vmatrix} = \left(\frac{9}{10}-\lambda\right)\left(\frac{1}{10}-\lambda\right) - \frac{9}{100}$$

$$= \frac{1}{100}\left((9-10\lambda)(1-10\lambda) - 9\right)$$

$$= \frac{1}{100}\left(9 - 90\lambda - 10\lambda + 100\lambda^2 - 9\right)$$

$$= \frac{1}{100}\left(-100\lambda + 100\lambda^2\right) = -\lambda + \lambda^2$$

$= \lambda(-1+\lambda) = 0$

$\Leftrightarrow \lambda = 0$ or $\lambda = 1$

These are precisely the eigenvalues we found last time.

Ex: Let $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$. Let's find the eigenvalues.

$\det(A - \lambda I) = \det\left(\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)$

$= \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 8$

$= (3 - \lambda - 3\lambda + \lambda^2) - 8 = \lambda^2 - 4\lambda - 5$

$= (\lambda - 5)(\lambda + 1) = 0$

$\Leftrightarrow \boxed{\lambda = 5 \text{ or } \lambda = -1}$

Ex: Let $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{bmatrix}$. Let's find the eigenvalues for A.

$$\det(A-\lambda I)=\begin{vmatrix} 2-\lambda & 3 & 4 \\ 0 & 3-\lambda & 4 \\ 0 & 0 & 4-\lambda \end{vmatrix}$$

$$=(2-\lambda)(3-\lambda)(4-\lambda)=0$$

$$\Leftrightarrow \lambda=2,\ \lambda=3 \text{ or } \lambda=4$$

Generally:

Theorem: If A is an upper (or lower) triangular matrix, with diagonal elements $a_1,\dots,a_n$, then the eigenvalues of A are $a_1,\dots,a_n$.

Generally the characteristic equ for a (2×2)-matrix $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$\det(A-\lambda I)=\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix}=(a-\lambda)(d-\lambda)-bc$$

$$=ad-a\lambda-d\lambda+\lambda^2-bc$$

$$= \lambda^2 - (a+d)\lambda + (ad - bc)$$

$$= \lambda^2 - tr(A)\lambda + \det(A)$$

Ex: Find all eigenvalues of

$$A = \begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 0 & 4 & 3 & 2 & 1 \\ 0 & 0 & 5 & 4 & 3 \\ 0 & 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 4 & 3 & 2 & 1 \\ 0 & 4-\lambda & 3 & 2 & 1 \\ 0 & 0 & 5-\lambda & 4 & 3 \\ 0 & 0 & 0 & 4-\lambda & 3 \\ 0 & 0 & 0 & 0 & 5-\lambda \end{vmatrix}$$

$$= (5-\lambda)^3 (4-\lambda)^2 = 0$$

$\lambda = 5$ or $\lambda = 4$.

This degree 5 polynomial has a triple root at $\lambda = 5$ & a double root

at $\lambda=4$.

In this case we say that the eigenvalue $\lambda=5$ has algebraic multiplicity 3. Similarly $\lambda=4$ has algebraic multiplicity 2.

Def: An eigenvalue λ of an $(n\times n)$-matrix A has algebraic multiplicity k if λ is a root of multiplicity k of the characteristic equation $\det(A-\lambda I)=0$.

Ex: Find all eigenvalues w/ their algebraic multiplicities of

$$A=\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}.$$

Characteristic equ: $\det(A-\lambda I)=$

$$= \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} \begin{matrix} \leftarrow \\ \\ \leftarrow \end{matrix} \begin{matrix} \\ -1 \\ \\ \end{matrix} \begin{matrix} \\ -(1-\lambda) \\ \\ \end{matrix} = \begin{vmatrix} 0 & 1-(1-\lambda)^2 & \lambda \\ 1 & 1-\lambda & 1 \\ 0 & \lambda & -\lambda \end{vmatrix}$$

$$= -\begin{vmatrix} 1-(1-\lambda)^2 & \lambda \\ \lambda & -\lambda \end{vmatrix} = -\left((1-(1-\lambda)^2)(-\lambda) - \lambda^2 \right)$$

$$= \lambda(1-(1-\lambda)^2) + \lambda^2 = \lambda\left(1-(1-2\lambda+\lambda^2)\right) + \lambda^2$$

$$= \lambda\left(2\lambda - \lambda^2 + \lambda\right) = \lambda^2(3-\lambda) = 0$$

So $\lambda = 0$ is an eigenvalue w/ algebraic multiplicity 2, and $\lambda = 3$ is an eigenvalue w/ alg. mul. 1.