§7.1 Diagonalization

Consider $A=\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}$, $B=\begin{bmatrix}1&2&3\\0&-1&0\\1&2&0\end{bmatrix}$

Let us compute powers of A and B.

$$A^2=A\cdot A=\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}=\begin{bmatrix}1&0&0\\0&4&0\\0&0&9\end{bmatrix}$$

$$A^3=A^2\cdot A=\begin{bmatrix}1&0&0\\0&4&0\\0&0&9\end{bmatrix}\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}=\begin{bmatrix}1&0&0\\0&8&0\\0&0&27\end{bmatrix}$$

2^3 3^3

Generally:

$$A^n=\begin{bmatrix}1^n&0&0\\0&2^n&0\\0&0&3^n\end{bmatrix}.$$

What about B^n?

$$B^2=\begin{bmatrix}1&2&3\\0&-1&0\\1&2&0\end{bmatrix}\begin{bmatrix}1&2&3\\0&-1&0\\1&2&0\end{bmatrix}=\begin{bmatrix}4&6&3\\0&1&0\\1&0&3\end{bmatrix}$$

$$B^3 = B^2B = \begin{bmatrix} 4 & 6 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 0 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 8 & 12 \\ 0 & -1 & 0 \\ 4 & 8 & 3 \end{bmatrix}$$

... no real pattern.

A is a <u>diagonal matrix</u>, and in fact if

$$A = \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}, \text{ then } A^k = \begin{bmatrix} a_1^k & & 0 \\ & \ddots & \\ 0 & & a_n^k \end{bmatrix}$$

for any positive integer k!
Computing powers of diagonal matrices is very easy & desirable in applications.

<u>Recall</u>: From 2.10/§3.4:

If $B = (\vec{v}_1, \ldots, \vec{v}_n)$ is a basis for $\mathbb{R}^n$, and if $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear transformation $T(\vec{x}) = A\vec{x}$ then the B-matrix for T is

$[A]_B = S^{-1}AS$, where $S = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$.

The B-matrix $[A]_B$

$$\boxed{[T(\vec{x})]_B = [A]_B [\vec{x}]_B}$$

$[\vec{x}]_B$ = B-coordinates of $\vec{x}$

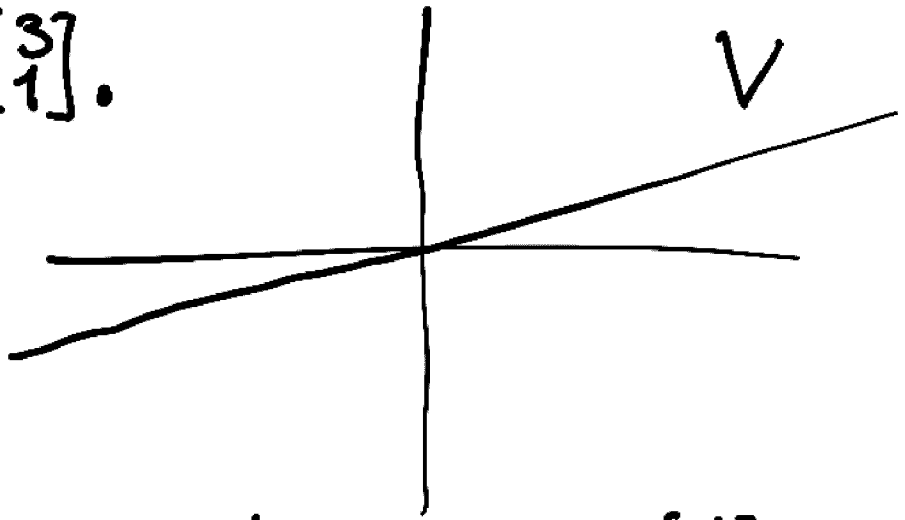
$[T(\vec{x})]_B$ = B-coordinates of $T(\vec{x})$

An equivalent way of thinking about this is that $[A]_B$ is "the matrix A expressed in the basis B".

Def: An $(n \times n)$-matrix A is diagonalizable if there exists an invertible matrix S such that $S^{-1}AS$ is a diagonal matrix.

Ex: $A = \frac{1}{10}\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$. This is the matrix for Proj_V where

V is the subspace of $\mathbb{R}^2$ spanned by $\begin{bmatrix}3\\1\end{bmatrix}$.



The two vectors $\vec{v}_1=\begin{bmatrix}3\\1\end{bmatrix}, \vec{v}_2=\begin{bmatrix}-1\\3\end{bmatrix}$ form a basis $B=\left(\begin{bmatrix}3\\1\end{bmatrix},\begin{bmatrix}-1\\3\end{bmatrix}\right)$ for $\mathbb{R}^2$ since $\dim \mathbb{R}^2=2$, and they are linearly independent (since they are not parallel).

Forming $S=\begin{bmatrix}3 & -1\\ 1 & 3\end{bmatrix}$ we may compute its inverse to be

$S^{-1}=\frac{1}{10}\begin{bmatrix}3 & 1\\ -1 & 3\end{bmatrix}$. We then compute the matrix A in the basis B:

$$[A]_B=S^{-1}AS=\frac{1}{100}\begin{bmatrix}3 & 1\\ -1 & 3\end{bmatrix}\begin{bmatrix}9 & 3\\ 3 & 1\end{bmatrix}\begin{bmatrix}3 & -1\\ 1 & 3\end{bmatrix}$$

$$= \frac{1}{100}\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 30 & 0 \\ 10 & 0 \end{bmatrix} = \frac{1}{100}\begin{bmatrix} 100 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

so $[A]_B$ is diagonal! Our original matrix is <u>not</u> diagonal, but it is <u>diagonalizable</u>.

<u>Def:</u> A linear transformation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $T(\vec{x}) = Ax$ is called <u>diagonalizable</u> if there exists a basis B for $\mathbb{R}^n$ such that $[A]_B$ is a diagonal matrix.

<u>Ex</u> In the previous example, we saw that $\text{proj}_V: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ where $V = \text{span}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, is diagonalizable, because if we choose $B = \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}\right)$, then $[A]_B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

So if $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is diagonalizable $T(\vec{x}) = A\vec{x}$

We can find a basis $B=(\vec{v}_1,\dots,\vec{v}_n)$ for $\mathbb{R}^n$ such that

$$[A]_B=\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}.$$

Recall $$A=\begin{bmatrix}| & & |\\ T(\vec{e}_1) & \cdots & T(\vec{e}_n)\\ | & & |\end{bmatrix}.$$

Similarly, $$[A]_B=\begin{bmatrix}| & & |\\ [T(\vec{v}_1)]_B & \cdots & [T(\vec{v}_n)]_B\\ | & & |\end{bmatrix}.$$

Note $\vec{v}_i=0\cdot\vec{v}_1+\cdots+0\cdot\vec{v}_{i-1}+\vec{v}_i+0\cdot\vec{v}_{i+1}+\cdots+0\cdot\vec{v}_n$

So $[\vec{v}_i]_B=\begin{bmatrix}0\\ \vdots\\ 0\\ 1\\ 0\\ \vdots\\ 0\end{bmatrix}$ ← position i.

So if $[A]_B$ is diagonal it means

$$[T(\vec{v}_i)]_B=\begin{bmatrix}0\\ \vdots\\ 0\\ \lambda_i\\ 0\\ \vdots\\ 0\end{bmatrix}=\lambda_i[\vec{v}_i]_B$$

$\Leftrightarrow S^{-1}T(\vec{v}_i) = \lambda_i S^{-1}\vec{v}_i$

$\Leftrightarrow \boxed{T(\vec{v}_i) = \lambda_i \vec{v}_i}$

We conclude:

Theorem: If $T: \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation $T(\vec{x}) = A\vec{x}$, and $\mathcal{B} = (\vec{v}_1, \ldots, \vec{v}_n)$ is a basis for $\mathbb{R}^n$, then the $\mathcal{B}$-matrix of T, $[A]_{\mathcal{B}}$, is diagonal if and only if $\boxed{T(\vec{v}_i) = \lambda_i \vec{v}_i}$ for some scalar λ_i, for all $1 \leq i \leq n$.

The idea to find out whether a matrix A is diagonalizable is to try to find a basis $\mathcal{B} = (\vec{v}_1, \ldots, \vec{v}_n)$ such that $T(\vec{v}_i) = \lambda_i \vec{v}_i$ for some λ_i, for all i.

Such vectors/scalars are important & they have names:

Def: Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, $T(\vec{x}) = A\vec{x}$.

A non-zero vector $\vec{v}$ is called an eigenvector of T (or A) if

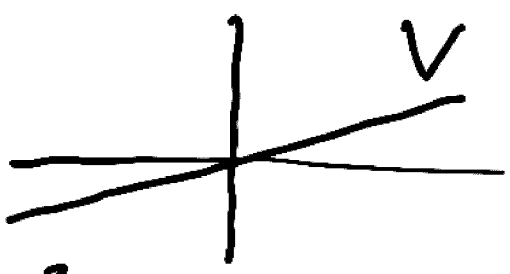
$T(\vec{v}) = \lambda\vec{v}$ (or $A\vec{v} = \lambda\vec{v}$)

for some scalar λ.

This scalar λ is called the eigenvalue associated with the eigenvector $\vec{v}$.

A basis $(\vec{v}_1, \ldots, \vec{v}_n)$ of $\mathbb{R}^n$ is called an eigenbasis for T (or A) if each basis vector $\vec{v}_j$ is an eigenvector of T (or A).

Ex: $V = \text{span}\begin{bmatrix}3\\1\end{bmatrix}$ in $\mathbb{R}^2$



We consider $\text{proj}_V : \mathbb{R}^2 \to \mathbb{R}^2$.

We found its matrix:

$A = \frac{1}{10}\begin{bmatrix}9 & 3\\3 & 1\end{bmatrix}$, and with respect to the basis $B = \left(\begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}-1\\3\end{bmatrix}\right)$, we found

$[A]_B = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$.

$\vec{v}_1$ $\vec{v}_2$

How does proj_V act on $\vec{v}_1$ and $\vec{v}_2$?

$$A\vec{v}_1 = \frac{1}{10}\begin{bmatrix}9 & 3\\3 & 1\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix} = \frac{1}{10}\begin{bmatrix}30\\10\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix} = \vec{v}_1$$

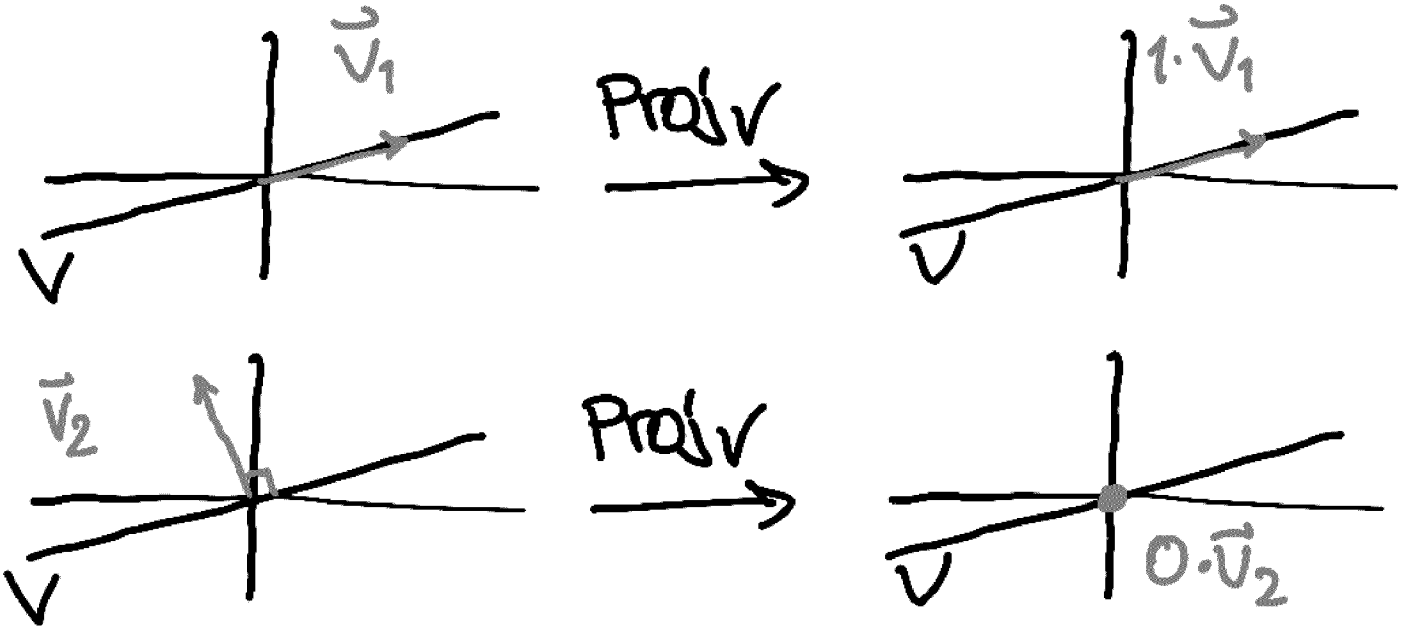
$$A\vec{v}_2 = \frac{1}{10}\begin{bmatrix}9 & 3\\3 & 1\end{bmatrix}\begin{bmatrix}-1\\3\end{bmatrix} = \frac{1}{10}\begin{bmatrix}0\\0\end{bmatrix} = 0\cdot\vec{v}_2$$

So $\vec{v}_1$ is an eigenvector with eigenvalue $\lambda_1 = 1$.

$\vec{v}_2$ is an eigenvector with eigenvalue $\lambda_2 = 0$

B is an eigenbasis for A.

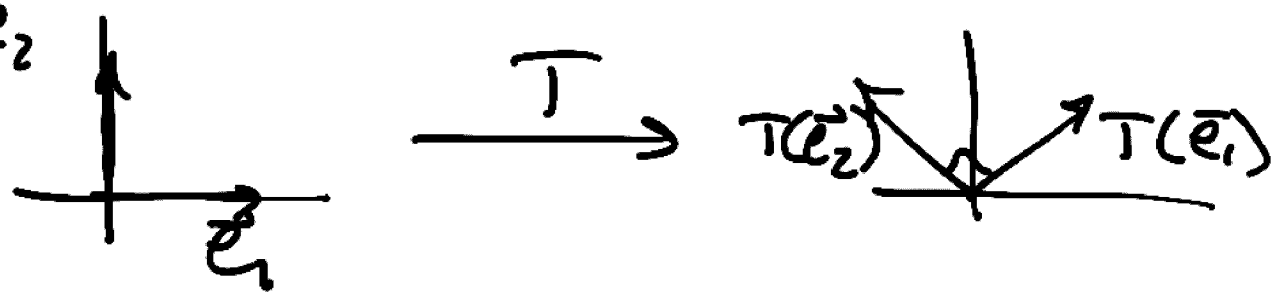
Geometrically:



Ex: Consider $I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

Since $I_n \vec{v} = \vec{v}$ for any $\vec{v}$ in $\mathbb{R}^n$, any vector is an eigenvector with eigenvalue 1.

Ex: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(\vec{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x}$.



T is rotation 90° counterclockwise.
What are the eigenvectors of T?
Meaning: is there some $\vec{v} \neq \vec{0}$ in $\mathbb{R}^2$ such that $T(\vec{v}) = \lambda \vec{v}$?
($T(\vec{v})$ parallel to $\vec{v}$)
The answer is intuitively <u>no</u>
because if we take any vector $\vec{v}$ & rotate it 90°, the result will never be parallel to $\vec{v}$.

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad \lambda\vec{v} = \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \end{bmatrix}$$

$$A\vec{v} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -v_2 \\ v_1 \end{bmatrix}$$

$$A\vec{v} = \lambda\vec{v} \iff \begin{bmatrix} -v_2 \\ v_1 \end{bmatrix} = \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \end{bmatrix}$$

$$\iff \begin{cases} v_2 = -\lambda v_1 \\ v_1 = \lambda v_2 \end{cases}$$

So $v_2 = -\lambda v_1 = -\lambda(\lambda v_2)$

$$\iff v_2 + \lambda^2 v_2 = 0 \iff v_2(1+\lambda^2) = 0 \quad (*)$$

If $v_2 = 0$, then it implies $v_1 = 0$ but we assume $\vec{v} \neq \vec{0}$, so this cannot happen. So (*) must imply $1 + \lambda^2 = 0$, which has no real solutions.

So $A\vec{v} = \lambda\vec{v}$ has no solutions & so T has no eigenvectors.

We conclude that T is not diagonalizable.