Recall: Determinants:

$A = (n \times n)$ -matrix always

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- A = upper (or lower) triangular then $\det A$ = product of the diagonal elements.

- If B is obtained from A by adding a multiple of a row to another row, then

$$\det B = \det A.$$

- Laplace expansion:

Along the j-th column:

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

a_{ij} = element at position (i,j) in A

A_{ij} = minor of A at position (i,j)

Along the i-th row:

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

Ex: $A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 9 & 1 & 3 & 0 \\ 9 & 2 & 2 & 0 \\ 5 & 0 & 0 & 3 \end{bmatrix}$ ← a_{14}, ← a_{44}

Let's compute $\det A$ via Laplace expansion. Let's pick the last column.

$$\det A = (-1)^5 \cdot 2 \cdot \underbrace{\begin{vmatrix} 9 & 1 & 3 \\ 9 & 2 & 2 \\ 5 & 0 & 0 \end{vmatrix}}_{\det(A_{14})} + (-1)^8 \cdot 3 \cdot \underbrace{\begin{vmatrix} 1 & 0 & 1 \\ 9 & 1 & 3 \\ 9 & 2 & 2 \end{vmatrix}}_{\det(A_{44})}$$

① ②

Now lets compute each of the (3x3)-determinants separately:

① $\begin{vmatrix} 9 & 1 & 3 \\ 9 & 2 & 2 \\ 5 & 0 & 0 \end{vmatrix}$ Laplace expansion along row 3 $= (-1)^4 \cdot 5 \cdot \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix}$

a_{31}

$= 5(2-6) = -20$

a_{11}

② $\begin{vmatrix} 1 & 0 & 1 \\ 9 & 1 & 3 \\ 9 & 2 & 2 \end{vmatrix}$ expand row 1 $= (-1)^2 \cdot 1 \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} + (-1)^4 \cdot 1 \cdot \begin{vmatrix} 9 & 1 \\ 9 & 2 \end{vmatrix}$

$= (2-6) + (18-9) = -4+9 = 5$

So $\det A = (-1)^5 \cdot 2 \cdot (-20) + (-1)^8 \cdot 3 \cdot 5$

$= 40 + 15 = 55.$

Laplace expansion becomes very efficient if a matrix contains many 0's. We can combine row operations with Laplace

expansion.

EX:
$$\begin{vmatrix} 1 & 0 & 1 & 2 \\ 9 & 1 & 3 & 0 \\ 9 & 2 & 2 & 0 \\ 5 & 0 & 0 & 3 \end{vmatrix} \begin{matrix} \\ \\ \\ \\ \end{matrix} \begin{matrix} \\ \\ \\ \\ \end{matrix} \begin{matrix} \\ -3 \\ \\ \\ \end{matrix} \begin{matrix} \\ \\ \\ \\ \end{matrix} \begin{matrix} \\ \\ \\ \\ \end{matrix} \begin{matrix} \\ -2 \\ \\ \\ \end{matrix} = \begin{vmatrix} 1 & 0 & 1 & 2 \\ 6 & 1 & 0 & -6 \\ 7 & 2 & 0 & -4 \\ 5 & 0 & 0 & 3 \end{vmatrix}$$

(a_{13}: the circled 1 in row 1, column 3)

$$= (-1)^4 \cdot 1 \cdot \begin{vmatrix} 6 & 1 & -6 \\ 7 & 2 & -4 \\ 5 & 0 & 3 \end{vmatrix}$$

(a_{31}: the 5 in row 3, column 1; a_{33}: the 3 in row 3, column 3)

$$= (-1)^4 \cdot 5 \cdot \begin{vmatrix} 1 & -6 \\ 2 & -4 \end{vmatrix} + (-1)^6 \cdot 3 \cdot \begin{vmatrix} 6 & 1 \\ 7 & 2 \end{vmatrix}$$

$$= 5(-4+12) + 3(12-7) = 40+15 = 55.$$

EX:
$$A = \begin{bmatrix} 1 & -3 & 1 & -2 \\ 2 & -5 & -1 & -2 \\ 0 & -4 & 5 & 1 \\ -3 & 10 & -6 & 8 \end{bmatrix}$$ compute $\det A$.

$$\begin{vmatrix} 1 & -3 & 1 & -2 \\ 2 & -5 & -1 & -2 \\ 0 & -4 & 5 & 1 \\ -3 & 10 & -6 & 8 \end{vmatrix} \begin{matrix} \\ \\ \\ \\ \end{matrix} \begin{matrix} \\ \\ \\ \\ \end{matrix} \begin{matrix} \\ -2 \\ \\ \\ \end{matrix} \begin{matrix} \\ \\ \\ \\ \end{matrix} \begin{matrix} \\ \\ \\ \\ \end{matrix} \begin{matrix} \\ 3 \\ \\ \\ \end{matrix} = \begin{vmatrix} 1 & -3 & 1 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & -4 & 5 & 1 \\ 0 & 1 & -3 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -3 & 2 \\ -4 & 5 & 1 \\ 1 & -3 & 2 \end{vmatrix} \begin{matrix} \\ \\ \leftarrow \end{matrix} \begin{matrix} \\ \\ +4 \end{matrix} \begin{matrix} \\ \\ \leftarrow \end{matrix} \begin{matrix} \\ \\ -1 \end{matrix} = \begin{vmatrix} 1 & -3 & 2 \\ 0 & -7 & 9 \\ 0 & 0 & 0 \end{vmatrix} \leftarrow = 0$$

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§6.3 Cramer's rule

If a (2×2)-matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible, we can express its inverse in terms of its determinant:

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad (*)$$

This can in fact be helpful to finding solutions to

$$\begin{cases} ax_1 + bx_2 = b_1 \\ cx_1 + dx_2 = b_2 \end{cases} \iff A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\iff \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \overset{(*)}{=} \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$= \frac{1}{\det A} \begin{bmatrix} db_1 - bb_2 \\ -cb_1 + ab_2 \end{bmatrix}$$

Notice

$$db_1 - bb_2 = \det\underbrace{\begin{bmatrix} b_1 & b \\ b_2 & d \end{bmatrix}}_{\text{Matrix } A \text{ but w/ } \underline{\text{first}} \text{ column replaced by } \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}$$

$$-cb_1 + ab_2 = \det\underbrace{\begin{bmatrix} a & b_1 \\ c & b_2 \end{bmatrix}}_{\text{Matrix } A \text{ but w/ } \underline{\text{second}} \text{ column replaced by } \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}$$

Generally let $A_{\vec{b},i}$ be the matrix A but with column i replaced by $\vec{b}$.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A_{\vec{b},1} = \begin{bmatrix} b_1 & b \\ b_2 & d \end{bmatrix}, \quad A_{\vec{b},2} = \begin{bmatrix} a & b_1 \\ c & b_2 \end{bmatrix}.$$

Solutions to $A\vec{x} = \vec{b}$ are then

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

$$x_1 = \frac{\det A_{\vec{b},1}}{\det A}, \quad x_2 = \frac{\det A_{\vec{b},2}}{\det A}.$$

This is called Cramer's rule & it works for $n \times n$ systems!

EX: Consider $\begin{cases} 2x_1 + 3x_2 = 7 \\ 4x_1 + 5x_2 = 13 \end{cases}$

& let's use Cramer's rule to solve it.

$$\begin{cases} 2x_1 + 3x_2 = 7 \\ 4x_1 + 5x_2 = 13 \end{cases} \Leftrightarrow \underbrace{\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 7 \\ 13 \end{bmatrix}}_{\vec{b}}$$

$$A_{\vec{b},1} = \begin{bmatrix} 7 & 3 \\ 13 & 5 \end{bmatrix}, \quad A_{\vec{b},2} = \begin{bmatrix} 2 & 7 \\ 4 & 13 \end{bmatrix}$$

$\det A = 10 - 12 = -2$

$\det A_{\vec{b},1} = 35 - 39 = -4$

$\det A_{\vec{b},2} = 26 - 28 = -2$

$$x_1 = \frac{-4}{-2} = 2, \quad x_2 = \frac{-2}{-2} = 1$$

Ex Solve the system

$$\begin{cases} (b-1)x_1 + a x_2 = 0 \\ -a x_1 + (b-1)x_2 = C \end{cases}$$

Where a, b, C are arbitrary constants.

$$A = \begin{bmatrix} b-1 & a \\ -a & b-1 \end{bmatrix}. \quad \det A = (b-1)^2 + a^2$$

$$x_1 = \frac{\begin{vmatrix} 0 & a \\ C & b-1 \end{vmatrix}}{\det A} = \frac{-aC}{(b-1)^2 + a^2}$$

$$x_2 = \frac{\begin{vmatrix} b-1 & 0 \\ -a & C \end{vmatrix}}{\det A} = \frac{(b-1)C}{(b-1)^2 + a^2}.$$

Ex: Let's use Cramer's rule to solve

$$\begin{cases} x + y = 3 \\ y + z = 1 \\ x + z = 0 \end{cases}$$

$$\vec{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 1 - (-1) = 2$$

$$\det A_{b,1} = \begin{vmatrix} 3 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 3 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 2$$

$$\det A_{b,2} = \begin{vmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 3 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= 1 - 3(-1) = 4$$

$$\det A_{b,3} = \begin{vmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = 1 - 3 = -2$$

$$x_1 = \frac{2}{2} = 1, \quad x_2 = \frac{4}{2} = 2, \quad x_3 = \frac{-2}{2} = -1$$

Classical adjoint : $A = (n \times n)$-matrix

There's a matrix called the classical adjoint of A, $\operatorname{adj}(A)$

So that if A is invertible, then

$$A^{-1} = \frac{1}{\det A}\,\mathrm{adj}(A) = \frac{1}{\det A}\,C^T$$

The matrix C at position (i,j) has the element

"cofactor matrix"

$$m_{ij} = (-1)^{i+j}\det(A_{ij})$$

A_{ji} = minor at position (j,i).

Ex: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$A_{11} = [d]$, $A_{12} = [c]$, $A_{21} = [b]$

$A_{22} = [a]$

$$\mathrm{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Ex: $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

Cofactor matrix:

$$C = \begin{bmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & \begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ \begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} ei-hf & gf-di & dh-ge \\ hc-bi & ai-gc & gb-ah \\ bf-ec & dc-af & ae-bd \end{bmatrix}$$

$\text{adj } A = C^T$ = transpose of