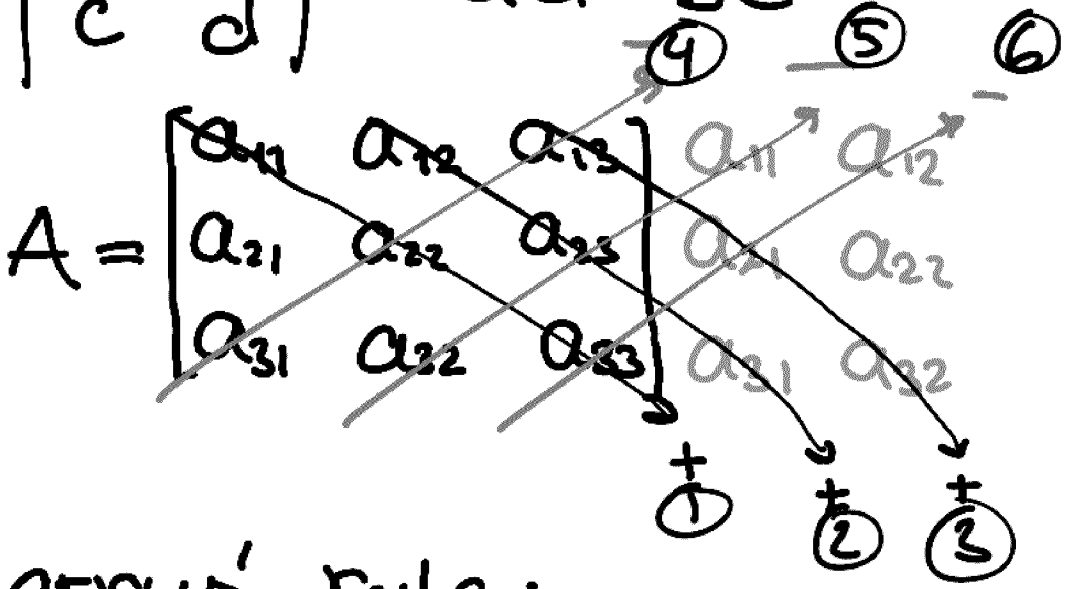
## Recall: Determinants:

$A = (n \times n)$ -matrix always

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$




Sarrus' rule:

$$\det A = \underset{①}{a_{11}a_{22}a_{32}} + \underset{②}{a_{12}a_{23}a_{31}} + \underset{③}{a_{13}a_{21}a_{32}} - \underset{④}{a_{31}a_{22}a_{13}} - \underset{⑤}{a_{32}a_{23}a_{11}} - \underset{⑥}{a_{33}a_{21}a_{12}}$$

## Four properties:

① $\det(A) \neq 0 \Leftrightarrow A$ is invertible

② $\det(A) = \det(A^T)$

③ $\det(AB) = \det(A)\det(B)$

④ $\det(kA) = k^n \det(A)$ $k$ = scalar, $A = (n \times n)$-matrix

**Theorem:** If $A$ is invertible, then

$$\det(A^{-1}) = \det(A)^{-1} = \frac{1}{\det A}.$$

---

**Proof:** By definition $AA^{-1} = I$

So

$$\det(AA^{-1}) = \det I = 1$$

$$\det(A)\det(A^{-1}) = 1 \Leftrightarrow \det(A^{-1}) = \frac{1}{\det(A)}. \quad \square$$

---

**Theorem:** If $A$ is orthogonal, then $\det(A) = \pm 1$.

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**Proof:** $A$ being orthogonal means $A^{-1} = A^T$, or equivalently, $AA^T = I$.

So

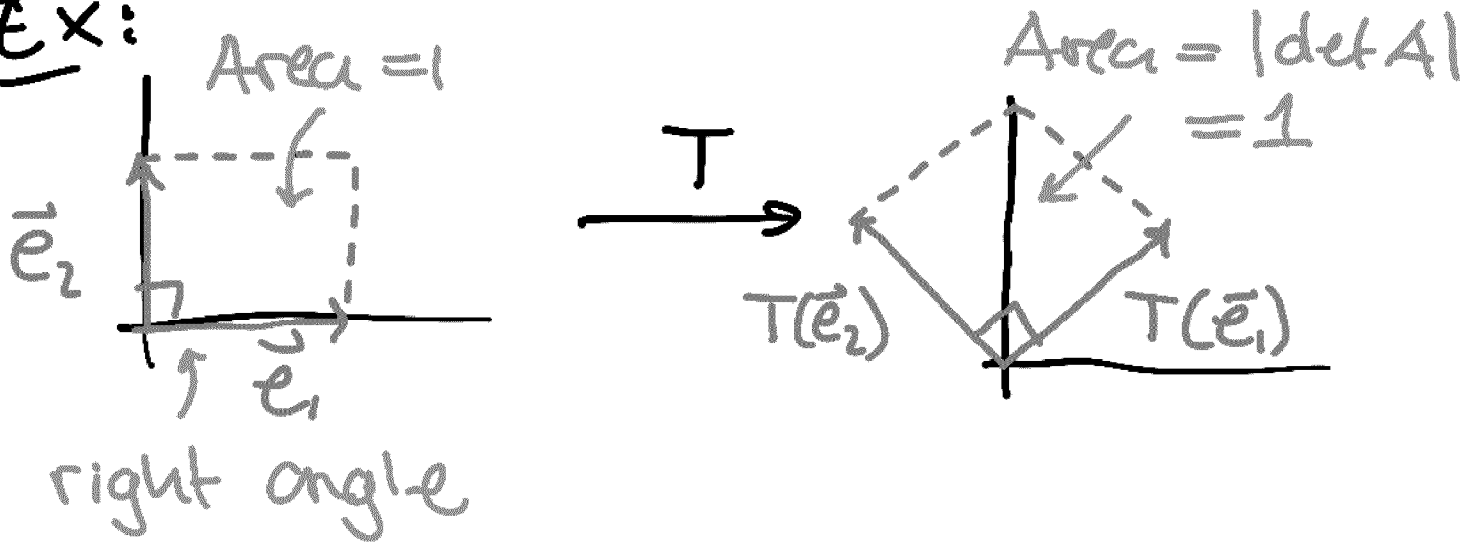
$$\det(AA^T) = \det(I) = 1$$

$$\Leftrightarrow \quad \det(A)\det(A^T) = 1$$

$$\Leftarrow \quad (\det(A))^2 = 1 \Leftrightarrow \det A = \pm 1 \quad \square$$

Rmk: Geometrically, orthogonal transformations $T: \mathbb{R}^n \to \mathbb{R}^n$ preserve angles and lengths, so for $n=2$:

Ex:




---

We only know how to compute determinants for 2x2 and (3x3 - maybe) so we finally need to discuss how to actually compute it in general.

First we will discuss/see how determinants can be computed via Gauss-Jordan elimination.

Theorem:

(1) If $A = \begin{bmatrix} — \vec{v}_1 — \\ — \vec{v}_i — \\ — \vec{v}_j — \\ \vdots \\ — \vec{v}_m — \end{bmatrix}$ $B = \begin{bmatrix} — \vec{v}_1 — \\ — \vec{v}_j — \\ — \vec{v}_i — \\ \vdots \\ — \vec{v}_m — \end{bmatrix}$

(i.e., B is obtained from A by swapping two rows)

then $\det B = -\det A$.

(2) If $A = \begin{bmatrix} \vdots \\ -\vec{v}_i- \\ \vdots \end{bmatrix}$, $B = \begin{bmatrix} \vdots \\ -k\vec{v}_i- \\ \vdots \end{bmatrix}$

(i.e., B is obtained from A by multiplying a row with the scalar k)

then $\det B = k \det A$

(3)

If $A = \begin{bmatrix} -\vec{v}_1- \\ -\vec{v}_j- \\ \vdots \end{bmatrix}$, $B = \begin{bmatrix} -\vec{v}_1- \\ -\vec{v}_j + k\vec{v}_i- \\ \vdots \end{bmatrix}$

(i.e., B is obtained from A by adding a multiple of a row to another row)

then $\det B = \det A$.

---

This helps us to compute $\det A$ in general!

Namely, if we perform Gauss-Jordan elimination $A \rightsquigarrow \text{rref}(A)$, and if we in this process

- swap rows $s$ times
- multiply various rows by the scalars $k_1, \ldots, k_r$

then

$$\det(A) = (-1)^s \frac{1}{k_1 \cdots k_r} \det(\text{rref}(A))$$

$\text{rref}(A)$ is by our algorithm an upper triangular matrix with

1's and 0's on the diagonal.

$\det(\text{rref}(A)) =$ product of diagonal elements

$= 0$ or $1$.

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$.

$k_1 = -\frac{1}{5}$

$\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \xrightarrow{-3} \begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix} \xrightarrow{\cdot(-\frac{1}{5})} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow{-2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \text{rref}(A)$.

- no row swaps, $s = 0$
- only multiplied one row w/ $k_1 = -\frac{1}{5}$.

$\det A = (-1)^0 k_1 \det(\text{rref}(A))$

$= (1)\left(\frac{1}{-1/5}\right)(1) = -5$.

Notice the shortcut! It's faster

to not do row operations all the way to rref(A) but to stop once the matrix is upper triangular & then use

$\det\begin{pmatrix}\text{upper}\\ \text{triangular}\end{pmatrix}$ = product of diagonal elements.

So:

$$A=\begin{bmatrix}1 & 2\\ 3 & 1\end{bmatrix}\overset{-3}{\longrightarrow}\begin{bmatrix}1 & 2\\ 0 & -5\end{bmatrix}=B$$

$\det A = \det B = (1)(-5) = -5.$

---

Let's revisit an example from last time:

Ex: $A=\begin{bmatrix}1 & -2 & -3\\ 1 & 1 & 1\\ 3 & 2 & 1\end{bmatrix}$ We computed

det by Sarrus' rule, but now let's do it w/ row operations.

$$A=\begin{bmatrix}1&-2&-3\\1&1&1\\3&2&1\end{bmatrix}\overset{-1}{\overset{-3}{\longrightarrow}}\begin{bmatrix}1&-2&-3\\0&3&4\\0&8&10\end{bmatrix}\overset{-\frac{8}{3}}{\longrightarrow}$$

$$\longrightarrow\begin{bmatrix}1&-2&-3\\0&3&4\\0&0&10-\frac{32}{3}\end{bmatrix}=B$$

$$\det A=\det B=(1)(3)\left(10-\tfrac{32}{3}\right)$$

$$=30-32=-2.$$

---

Ex: $A=\begin{bmatrix}1&0&1&2\\9&1&3&0\\9&2&2&0\\5&0&0&3\end{bmatrix}$ Let's compute $\det A$ by doing row ops.

$$A=\begin{bmatrix}1&0&1&2\\9&1&3&0\\9&2&2&0\\5&0&0&3\end{bmatrix}\overset{-9}{\overset{-9}{\overset{-5}{\longrightarrow}}}\begin{bmatrix}1&0&1&2\\0&1&-6&-18\\0&2&-7&-18\\0&0&-5&-7\end{bmatrix}\overset{-2}{\longrightarrow}$$

$$\longrightarrow\begin{bmatrix}1&0&1&2\\0&1&-6&-18\\0&0&5&18\\0&0&-5&-7\end{bmatrix}\overset{(+1)}{\longrightarrow}\begin{bmatrix}1&0&1&2\\0&1&-6&-18\\0&0&5&18\\0&0&0&11\end{bmatrix}=B$$

$\det A = \det B = (1)(1)(5\times(11) = 55.$

---

Doing many row operations to large matrices can also be time-consuming even though we just stop halfway.

## Minors & Laplace expansion

The most efficient way to compute determinants combines row operations with so-called "Laplace expansion".

**Def** (Minor of a matrix)
Let $A = (n\times n)$-matrix. For any $1 \le i, j \le n$, $A_{ij}$ is the $((n-1)\times(n-1))$-matrix obtained from $A$ by removing the $i$-th row & $j$-th column.

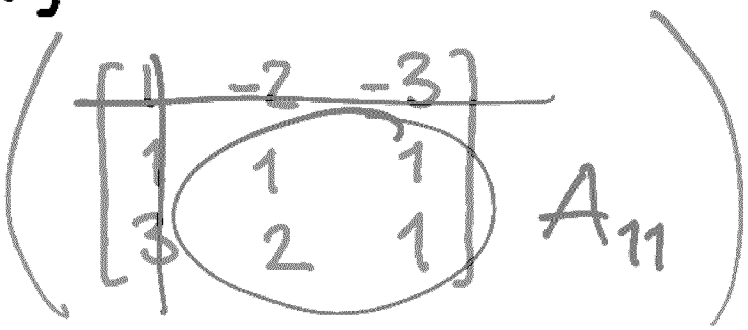
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**Ex:** $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $A_{11} = [4]$

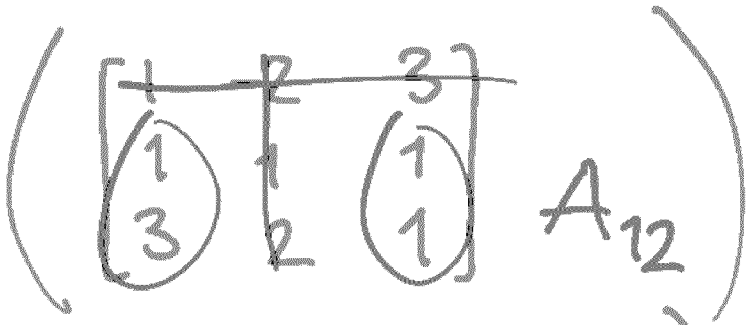
$A_{12} = [3]$, $A_{21} = [2]$, $A_{22} = [1]$.

Ex: $A = \begin{bmatrix} 1 & -2 & -3 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$

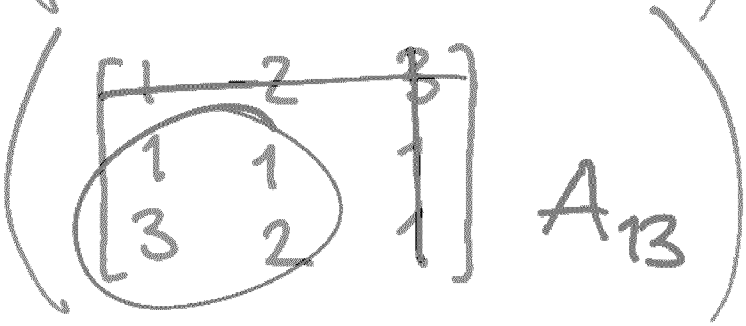
$A_{11} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$




$A_{12} = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$




$A_{13} = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$




⋮

etc.

---

**Theorem** (Laplace expansion)

Let $A$ be an $(n \times n)$-matrix.

Then we can either expand along the $j$-th column (for any $j$):

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

or expand along the $i$-th row (for any $i$):

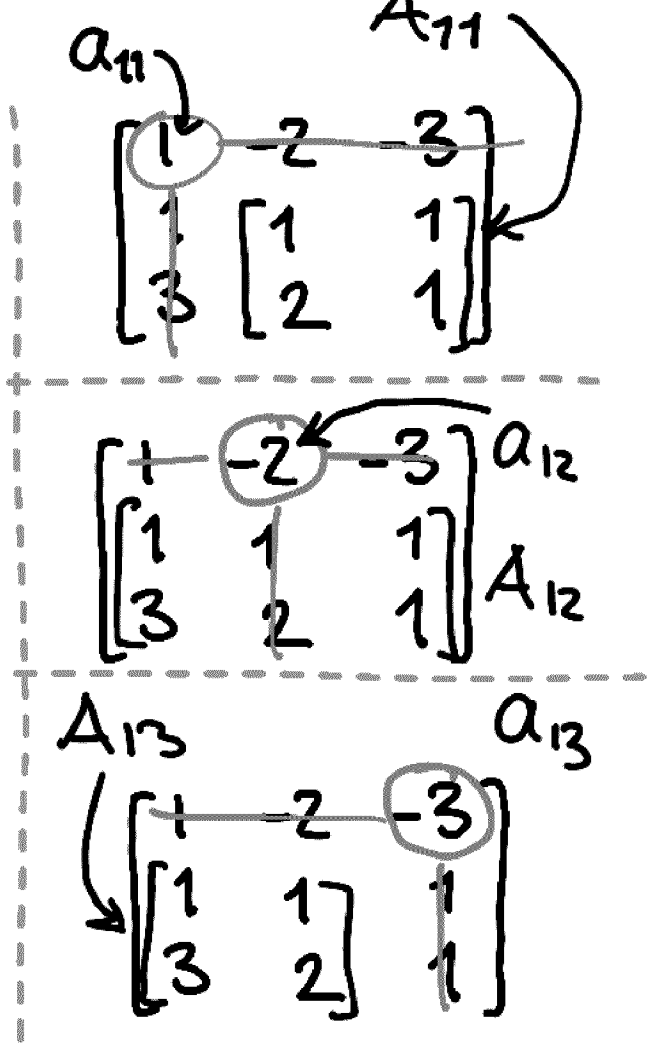
$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

---

EX: $A = \begin{bmatrix} 1 & -2 & -3 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$.

Let's expand along 1st row

$$\det A = (-1)^{1+1} a_{11} \det(A_{11}) + (-1)^{1+2} a_{12} \det(A_{12}) + (-1)^{1+3} a_{13} \det(A_{13})$$



$$= (1)\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} + (-3)\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= (1-2) + 2(1-3) - 3(2-3)$$

$$= (-1) - 4 + 3 = -2.$$

As an exercise you should try to perform Laplace expansion along any other row or column and verify that you get the same answer!

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