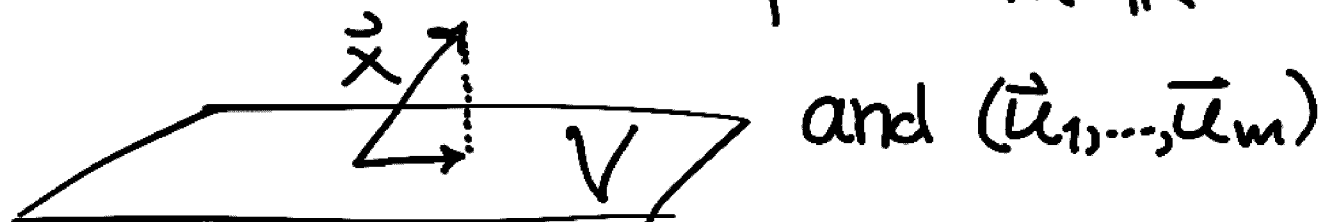


Recall: • A basis $(\vec{u}_1, \dots, \vec{u}_m)$ of a subspace V is orthonormal (ON) if $\|\vec{u}_i\| = 1$ and $\vec{u}_i \cdot \vec{u}_j = 0$ if $i \neq j$.

• If V is a subspace in $\mathbb{R}^n$



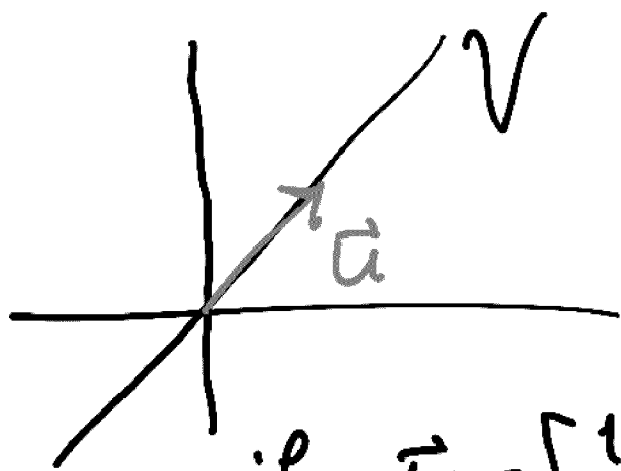
is an orthonormal basis for V , then the projection of a vector $\vec{x}$ onto V is

$$\text{proj}_V(\vec{x}) = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \dots + (\vec{u}_m \cdot \vec{x})\vec{u}_m.$$

§ 5.2 Gram-Schmidt process

In the projection formula we need to have an orthonormal basis for V .

Ex: $V =$ line in the plane



If $V = \text{graph of } y=x$
let's say, then

if $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $(\vec{u})$ is a basis for V . It clearly spans V & is linearly independent.

Is $(\vec{u})$ an orthonormal basis?

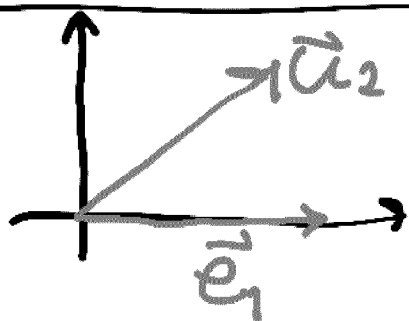
$$\|\vec{u}\| = \sqrt{1^2 + 1^2} = \sqrt{2} \neq 1 \quad \underline{\text{no}}$$

To get an ON basis, we need to normalize $\vec{u}$ to have length 1.

$$\text{Let } \vec{v} = \frac{\vec{u}}{\|\vec{u}\|} = \frac{\vec{u}}{\sqrt{2}} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}. \text{ Then}$$

we have $\|\vec{v}\| = 1$, so $(\vec{v})$ is an ON basis for $\vec{v}$.

Ex:



Consider

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The two vectors are linearly indep.
Since they are not parallel.

So $(\vec{e}_1, \vec{u}_2)$ is a basis for $\mathbb{R}^2$

Since $\dim \mathbb{R}^2 = 2$.

Is it an ON-basis?

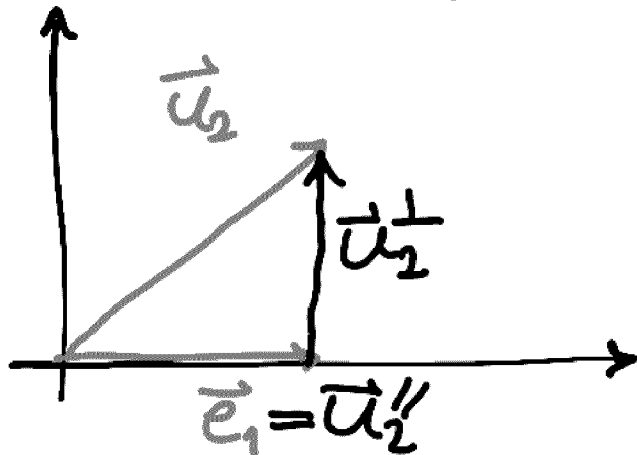
$$\|\vec{e}_1\| = \sqrt{1+0} = 1, \quad \|\vec{u}_2\| = \sqrt{1+1} = \sqrt{2}$$

$$\vec{e}_1 \cdot \vec{u}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1$$

NO, $\vec{u}_2$ isn't a unit vector &

$\vec{e}_1, \vec{u}_2$ are not orthogonal.

The projection formula can help
us transform $(\vec{e}_1, \vec{u}_2)$ to an ON-
basis however!



Let $V = x\text{-axis}$ (spanned by $\vec{e}_1$).

$$\vec{u}_2'' = \text{proj}_V(\vec{u}_2) = (\vec{e}_1 \cdot \vec{u}_2) \vec{e}_1 = \vec{e}_1$$

$$\vec{u}_2 = \vec{u}_2'' + \vec{u}_2^\perp$$

$$\Leftrightarrow \vec{u}_2^\perp = \vec{u}_2 - \vec{u}_2'' = \vec{u}_2 - \vec{e}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Change $\vec{u}_2$ to $\vec{u}_2^\perp \rightsquigarrow (\vec{e}_1, \vec{u}_2^\perp)$.

Then $\vec{e}_1 \cdot \vec{u}_2^\perp = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$
so they are orthogonal &

$$\|\vec{u}_2^\perp\| = \sqrt{0+1} = 1.$$

Therefore $(\vec{e}_1, \vec{u}_2^\perp)$ is an ON-basis for $\mathbb{R}^2$.

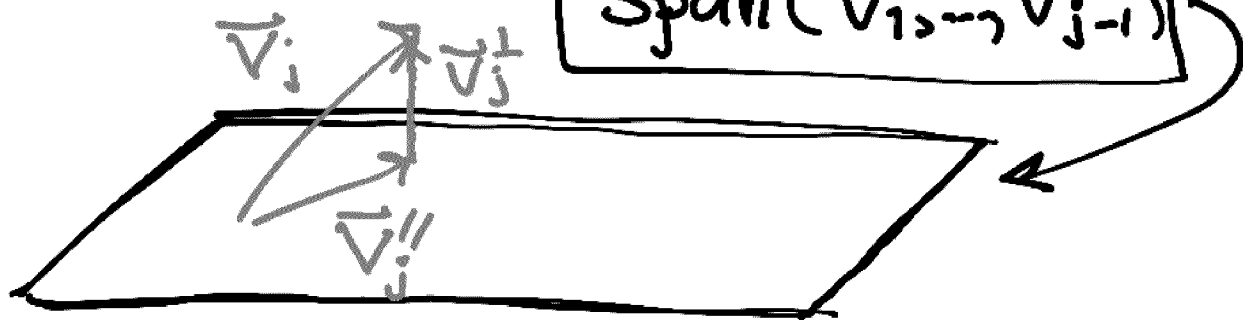
This example illustrates the idea how to transform a basis of V to an ON-basis. This process is called the Gram-Schmidt process.

The Gram-Schmidt process :

Consider a basis $\vec{v}_1, \dots, \vec{v}_m$ of a subspace V of $\mathbb{R}^n$.

For every $j=2, \dots, m$ write

$$\vec{v}_j = \vec{v}_j^\perp + \vec{v}_j'' \quad \text{with respect to} \quad \boxed{\text{Span}(\vec{v}_1, \dots, \vec{v}_{j-1})}$$



Then

$$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1, \quad \vec{u}_2 = \frac{1}{\|\vec{v}_2^\perp\|} \vec{v}_2^\perp$$

$$, \dots, \vec{u}_m = \frac{1}{\|\vec{v}_m^\perp\|} \vec{v}_m^\perp$$

is an ON-basis of V . We have

$$\begin{aligned} \vec{v}_j^\perp = \vec{v}_j - \vec{v}_j'' = \vec{v}_j - (\vec{u}_1 \cdot \vec{v}_j) \vec{u}_1 \\ - \dots - (\vec{u}_{j-1} \cdot \vec{v}_j) \vec{u}_{j-1} \end{aligned}$$

Ex: Consider the subspace

$V = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 9 \\ 9 \\ 1 \end{bmatrix}\right)$ of $\mathbb{R}^4$, and find an ON-basis of V .

Sol: $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 9 \\ 9 \\ 1 \end{bmatrix}$ forms a basis for V since they are linearly indep.

Now we have by Gram-Schmidt:

$$\|\vec{v}_1\| = \sqrt{1+1+1+1} = 2, \text{ So}$$

$$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$\vec{v}_2 = \vec{v}_2'' + \vec{v}_2^\perp \Leftrightarrow \vec{v}_2^\perp = \vec{v}_2 - \vec{v}_2''$$

$$= \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2) \vec{u}_1$$

$$= \begin{bmatrix} 1 \\ 9 \\ 9 \\ 1 \end{bmatrix} - \left(\frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 9 \\ 9 \\ 1 \end{bmatrix} \right) \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 9 \\ 9 \\ 1 \end{bmatrix} - \frac{1}{4} (1+9+9+1) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 9 \\ 9 \\ 1 \end{bmatrix} - 5 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 4 \\ -4 \end{bmatrix}$$

$$\|\vec{v}_2\| = \sqrt{16+16+16+16} = \sqrt{64} = 8$$

$$\vec{u}_2 = \frac{1}{\|\vec{v}_2\|} \vec{v}_2 = \frac{1}{8} \begin{bmatrix} -4 \\ 4 \\ 4 \\ -4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

So $(\vec{u}_1, \vec{u}_2) = \left(\frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right)$ is an ON-basis for V .

Ex: Consider the basis

$$(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \left(\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} \right) \text{ of } \mathbb{R}^3.$$

Let's apply the Gram-Schmidt process to turn it into an ON-basis.

$$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1 = \frac{1}{\sqrt{4+0+0}} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{aligned} \vec{v}_2^\perp &= \vec{v}_2 - \vec{v}_2'' = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2) \vec{u}_1 \\ &= \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} - 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} \end{aligned}$$

$$\vec{u}_2 = \frac{1}{\|\vec{v}_2\|} \vec{v}_2^\perp = \frac{1}{\sqrt{0+16+0}} \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \vec{v}_3^\perp &= \vec{v}_3 - \vec{v}_3'' = \vec{v}_3 - (\vec{u}_1 \cdot \vec{v}_3) \vec{u}_1 - (\vec{u}_2 \cdot \vec{v}_3) \vec{u}_2 \\ &= \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} \right) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} - 5 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 6 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix} \end{aligned}$$

$$\vec{u}_3 = \frac{1}{\|\vec{v}_3^\perp\|} \vec{v}_3^\perp = \frac{1}{\sqrt{0+0+49}} \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

So applying the GS-process to

$$(\vec{v}_1, \vec{u}_2, \vec{v}_3) \text{ gives } (\vec{u}_1, \vec{u}_2, \vec{u}_3) = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

Which is an ON-basis for $\mathbb{R}^3$
(it's the standard basis).

§ 5.3 Orthogonal transformations

Before we discuss orthogonal transformations, let's discuss the following:

Pythagorean theorem :

For any two orthogonal vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$ we have

$$\|\vec{x}\|^2 + \|\vec{y}\|^2 = \|\vec{x} + \vec{y}\|^2.$$

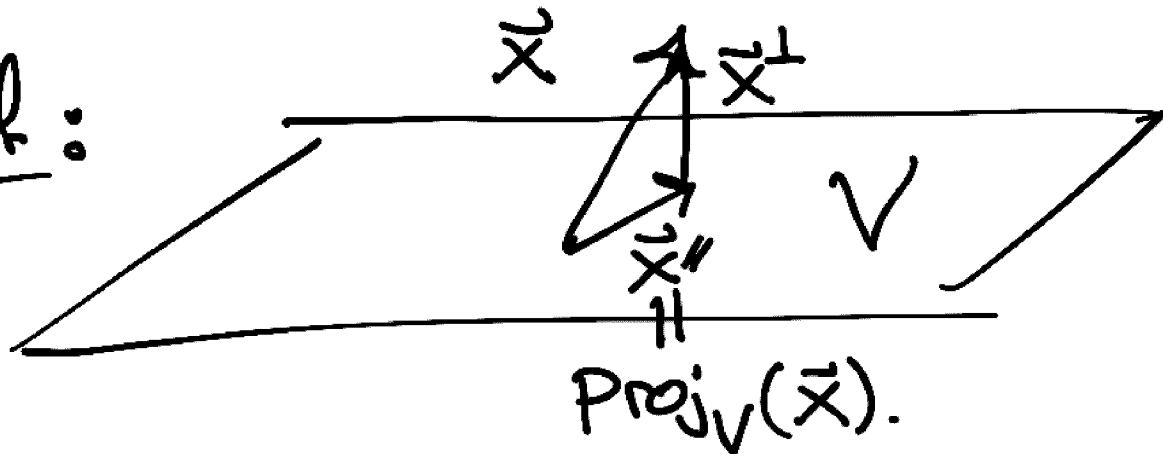
The proof is a calculation, since $\|\vec{x}\|^2 = \vec{x} \cdot \vec{x}$:

$$\begin{aligned}\|\vec{x} + \vec{y}\|^2 &= (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) \\ &= \underbrace{\vec{x} \cdot \vec{x}}_{=0} + \underbrace{\vec{y} \cdot \vec{x}}_{=0} + \underbrace{\vec{x} \cdot \vec{y}}_{=0} + \vec{y} \cdot \vec{y} \\ &\quad \text{Since } \vec{x} \text{ and } \vec{y} \text{ are orthogonal!} \\ &= \|\vec{x}\|^2 + \|\vec{y}\|^2.\end{aligned}$$

Theorem: If V is a subspace in $\mathbb{R}^n$ then for any vector $\vec{x}$ in $\mathbb{R}^n$

$$\|\text{Proj}_V(\vec{x})\| \leq \|\vec{x}\|.$$

Proof:



$$\vec{x} = \vec{x}'' + \vec{x}^\perp = \text{Proj}_V(\vec{x}) + \vec{x}^\perp$$

$\text{Proj}_V(\vec{x})$ and $\vec{x}^\perp$ are orthogonal
so Pythagorean theorem gives

$$\|\vec{x}\|^2 = \|\text{Proj}_V(\vec{x})\|^2 + \|\vec{x}^\perp\|^2$$

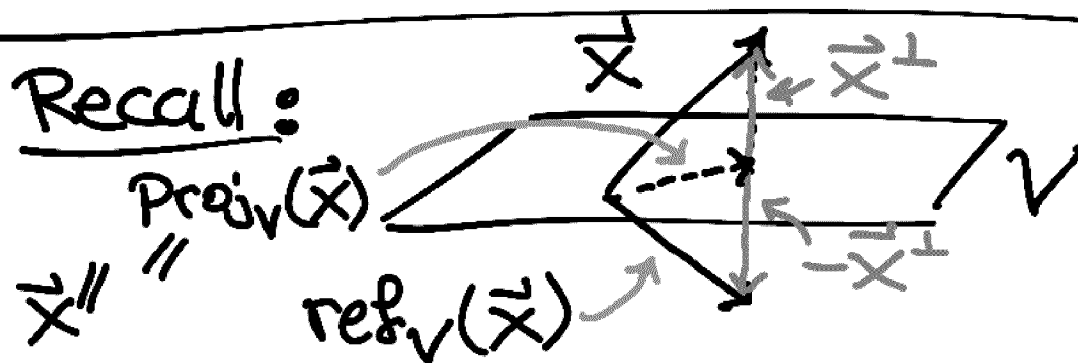
$$\|\vec{x}^\perp\|^2 \geq 0 \quad \text{so}$$

$$\|\text{Proj}_V(\vec{x})\|^2 + \|\vec{x}^\perp\|^2 \geq \|\text{Proj}_V(\vec{x})\|^2$$
$$\parallel$$
$$\|\vec{x}\|^2$$

$$\Rightarrow \|\text{Proj}_V(\vec{x})\| \leq \|\vec{x}\|.$$

□

Recall:



$$\text{ref}_V(\vec{x}) = \vec{x}'' - \vec{x}^\perp$$

Theorem: If V is a subspace in $\mathbb{R}^n$ then

$$\|\text{ref}_V(\vec{x})\| = \|\vec{x}\|$$

Proof: $\text{ref}_V(\vec{x}) = \vec{x}'' - \vec{x}^\perp = \vec{x}'' + (-\vec{x}^\perp)$

$\vec{x}''$ and $-\vec{x}^\perp$ are orthogonal, so Pythagorean theorem gives

$$\begin{aligned}\|\text{ref}_V(\vec{x})\|^2 &= \|\vec{x}''\|^2 + \|-\vec{x}^\perp\|^2 \\ &= \|\vec{x}''\|^2 + \|\vec{x}^\perp\|^2 = \|\vec{x}\|^2\end{aligned}$$

Since $\vec{x} = \vec{x}'' + \vec{x}^\perp$.

□
