MAT 211 L11 Wed Sep 30

Recall: • Dot product of two vectors

$$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \text{ and } \vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \text{ in } \mathbb{R}^n$$

$$\vec{v} \cdot \vec{w} = v_1 w_1 + \cdots + v_n w_n$$

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§5.1 Orthogonal projections

Have already gotten a glimpse of what orthogonal projections look like back in Lec 4.

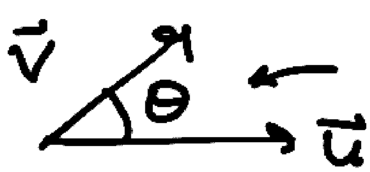
Def: (a) Two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ are orthogonal if $\vec{v} \cdot \vec{w} = 0$

(b) The length of $\vec{v}$ is

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + \cdots + v_n^2}$$

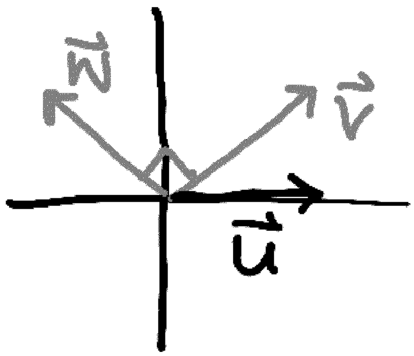
(c) A vector $\vec{u}$ is called a unit vector if $\|\vec{u}\| = 1$.

(d) We have $\vec{v} \cdot \vec{u} = \|\vec{v}\| \cdot \|\vec{u}\| \cdot \cos\theta$



← Smallest angle between $\vec{u}$, $\vec{v}$

Ex $\vec{v} = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{w} = \begin{bmatrix}-1\\1\end{bmatrix}$, $\vec{u} = \begin{bmatrix}1\\0\end{bmatrix}$.



$$\vec{v} \cdot \vec{w} = 1 \cdot (-1) + 1 \cdot 1 = 0$$
$$\vec{v} \cdot \vec{u} = 1 \cdot 1 + 1 \cdot 0 = 1$$
$$\vec{w} \cdot \vec{u} = (-1) 1 + 1 \cdot 0 = -1$$

$$\begin{cases} \|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{1 \cdot 1 + 1 \cdot 1} = \sqrt{2} \\ \|\vec{w}\| = \sqrt{(-1)^2 + 1^2} = \sqrt{2} \\ \|\vec{u}\| = \sqrt{1^2 + 0^2} = 1 \text{ , } \vec{u} \text{ unit vector} \end{cases}$$

Note we can scale any vector $\vec{v}$ to be a unit vector:

$\frac{1}{\|\vec{v}\|} \vec{v}$ is a unit vector

If V is a subspace in $\mathbb{R}^n$, we say that $\vec{w}$ is orthogonal to V if $\vec{w} \cdot \vec{v} = 0$ for all vectors $\vec{v}$ in V.

If we pick a basis $(\vec{v}_1, \dots, \vec{v}_m)$ for V, then to check if $\vec{w}$ is orthogonal to V it's enough to check $\vec{w} \cdot \vec{v}_i = 0$ for all $1 \leq i \leq m$.

<u>Ex</u>: Let $V = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right)$.

Is $\vec{w} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ orthogonal to V?

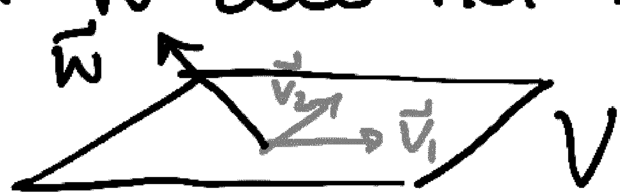
First, $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ form a basis for V because they span V, and they are linearly independent.

$$\vec{w} \cdot \vec{v}_1 = 1 \cdot (-1) + (-1) \cdot 1 + 0 \cdot 0 + 0 \cdot 1 = 0$$

$$\vec{w} \cdot \vec{v}_2 = 1 \cdot 0 + (-1) \cdot 1 + 0 \cdot 1 + 0 \cdot 1 = -1$$

So $\vec{w}$ is not orthogonal to V.

By calculation one can also check that $\vec{w}$ does not belong to V.



$\vec{w}$ doesn't lie inside V and also doesn't make a right angle with V.

<u>Def</u>: The vectors $\vec{v}_1, \ldots, \vec{v}_m$ in $\mathbb{R}^n$ are called orthonormal if they are all unit vectors and orthogonal to one another:

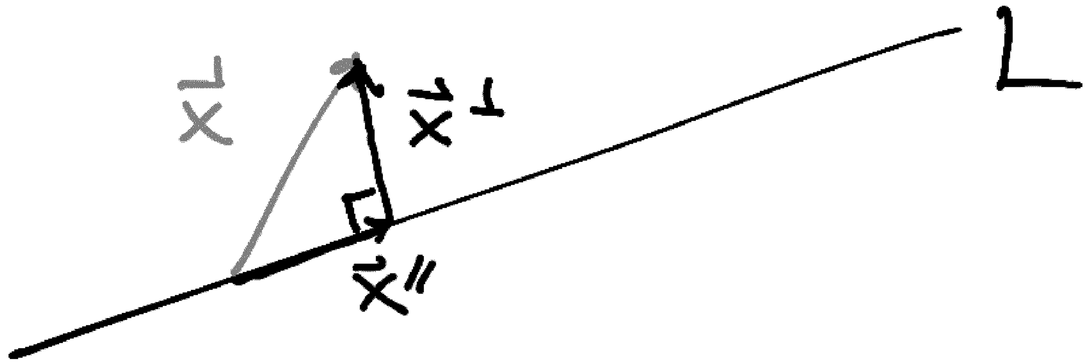
$$\vec{v}_i \cdot \vec{v}_j = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

<u>Ex</u>: $\vec{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$ are orthonormal because $\|\vec{e}_1\| = 1 = \|\vec{e}_2\|$ and $\vec{e}_1 \cdot \vec{e}_2 = 0$.

The following can be useful:

<u>Thm</u>: Orthonormal vectors are linearly independent.

Now let's discuss projections



The vector $\vec{x}$ can be written as a sum

$$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$$

parallel to L

orthogonal to L

The vector $\vec{x}^{\parallel}$ is called the <u>orthogonal projection of $\vec{x}$ onto</u> L

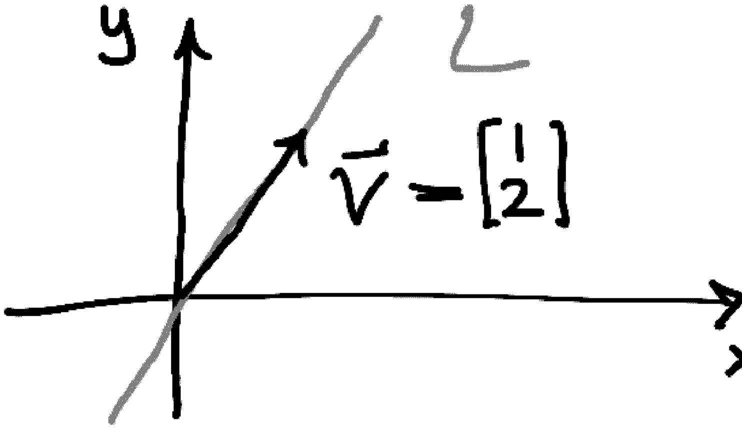
$$\vec{x}^{\parallel} = \mathrm{proj}_L(\vec{x})$$

<u>Theorem</u>: If V is a subspace with <u>orthonormal</u> basis $\vec{u}_1, \dots, \vec{u}_m$ then

$$\mathrm{proj}_V(\vec{x}) = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_m \cdot \vec{x})\vec{u}_m .$$

is the orthogonal projection of $\vec{x}$ onto V.

<u>Ex:</u> $L = \text{Span}\left(\begin{bmatrix}1\\2\end{bmatrix}\right)$ in $\mathbb{R}^2$



Let's find the orthogonal projection onto L of the vectors

$\vec{x}_1 = \begin{bmatrix}-2\\1\end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$ and $\vec{x}_3 = \begin{bmatrix}2\\2\end{bmatrix}$.

First have to find an orthonormal basis for L:

$L = \text{Span}(\vec{v})$, $\vec{v} = \begin{bmatrix}1\\2\end{bmatrix}$

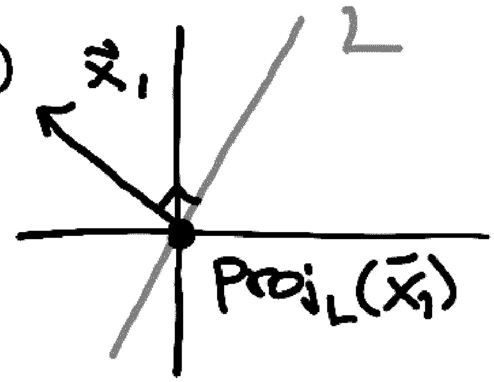
$\|\vec{v}\| = \sqrt{1^2+2^2} = \sqrt{5} \neq 1$ so $\vec{v}$ is not a unit vector. Define

$\vec{u} = \frac{1}{\sqrt{5}}\vec{v} = \frac{1}{\sqrt{5}}\begin{bmatrix}1\\2\end{bmatrix}$.

$L = \text{Span}(\vec{u})$, and so $\vec{u}$ is an orthonormal basis for L.

$$\operatorname{Proj}_L(\vec{x}_1) = (\vec{u}\cdot\vec{x}_1)\,\vec{u} = \left(\frac{1}{\sqrt{5}}\begin{bmatrix}1\\2\end{bmatrix}\cdot\begin{bmatrix}-2\\1\end{bmatrix}\right)\frac{1}{\sqrt{5}}\begin{bmatrix}1\\2\end{bmatrix}$$

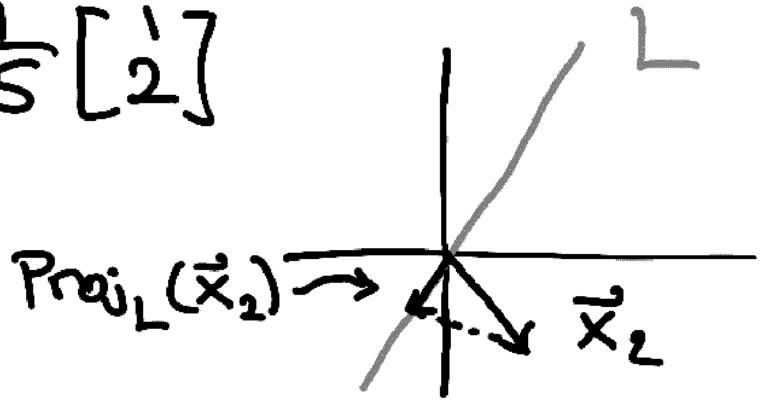
$$= \frac{1}{5}\underbrace{\left(1(-2)+2\cdot 1\right)}_{=0}\begin{bmatrix}1\\2\end{bmatrix} = \vec{0}$$



$$\operatorname{Proj}_L(\vec{x}_2) = \left(\frac{1}{\sqrt{5}}\begin{bmatrix}1\\2\end{bmatrix}\cdot\begin{bmatrix}1\\-1\end{bmatrix}\right)\frac{1}{\sqrt{5}}\begin{bmatrix}1\\2\end{bmatrix}$$

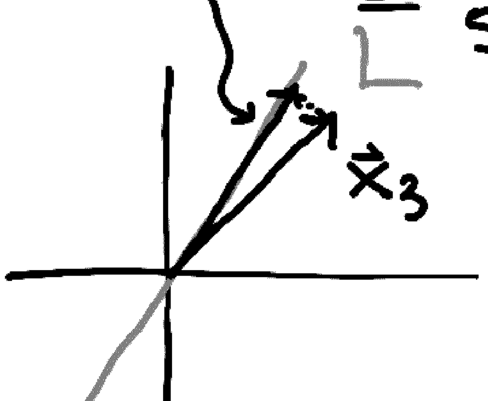
$$= \frac{1}{5}\left(1\cdot 1+2(-1)\right)\begin{bmatrix}1\\2\end{bmatrix}$$

$$= -\frac{1}{5}\begin{bmatrix}1\\2\end{bmatrix}$$



$$\boxed{\operatorname{Proj}_L(\vec{x}_3)} = \left(\frac{1}{\sqrt{5}}\begin{bmatrix}1\\2\end{bmatrix}\cdot\begin{bmatrix}2\\2\end{bmatrix}\right)\frac{1}{\sqrt{5}}\begin{bmatrix}1\\2\end{bmatrix}$$

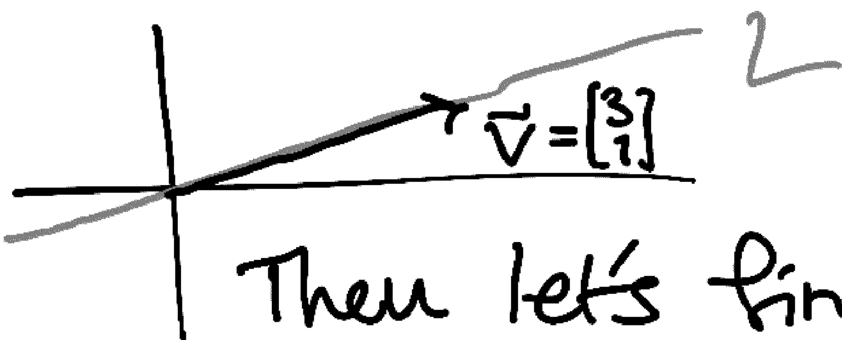
$$= \frac{1}{5}\left(1\cdot 2+2\cdot 2\right)\begin{bmatrix}1\\2\end{bmatrix} = \frac{6}{5}\begin{bmatrix}1\\2\end{bmatrix}$$



If V is a subspace of $\mathbb{R}^n$, then the <u>orthogonal complement</u> of V consists of all vectors that are orthogonal to V.

$V^\perp$ = orthogonal complement of V.

<u>Ex</u>: If $L = \text{span}\left(\begin{bmatrix}3\\1\end{bmatrix}\right)$



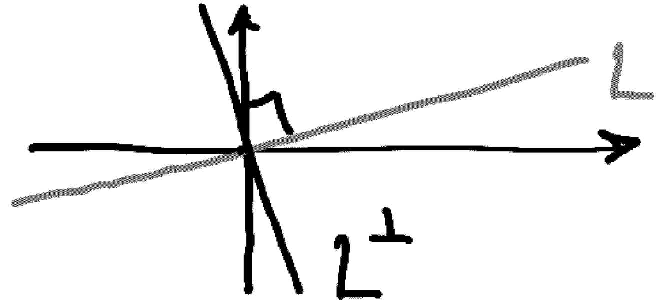
Then let's find $L^\perp$. It consists of all vectors that are orthogonal to L.

$\vec{x} = \begin{bmatrix}x\\y\end{bmatrix}$. $\vec{x}$ being orthogonal to L, means $\vec{x}\cdot\vec{v} = 0 \Leftrightarrow \begin{bmatrix}x\\y\end{bmatrix}\cdot\begin{bmatrix}3\\1\end{bmatrix} = 0$

$\Leftrightarrow 3x + y = 0 \Leftrightarrow x = t,\ y = -3x = -3t$

so all such vectors are

$$\vec{x} = \begin{bmatrix} t \\ -3t \end{bmatrix} = t\begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$



$$L^{\perp} = \text{span}\left(\begin{bmatrix} 1 \\ -3 \end{bmatrix}\right)$$