MAT 211 L10 Mon Sep 28

Recall: $\vec{v}_1, \dots, \vec{v}_m$ basis for subspace V in $\mathbb{R}^n$ if

- $\vec{v}_1, \dots, \vec{v}_m$ linearly independent (there is no redundant vector)
- $\vec{v}_1, \dots, \vec{v}_m$ spans V.

§3.4 Coordinates

Ex: Consider $\vec{v}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix}1\\2\\3\end{bmatrix}$ and the vector $\vec{x} = \begin{bmatrix}5\\7\\9\end{bmatrix}$.

The two vectors $\vec{v}_1$ and $\vec{v}_2$ span a plane V in $\mathbb{R}^3$.

Let's now check if $\vec{x}$ belongs to the plane.

In other words we are asking if there are scalars c_1, c_2 s.th.

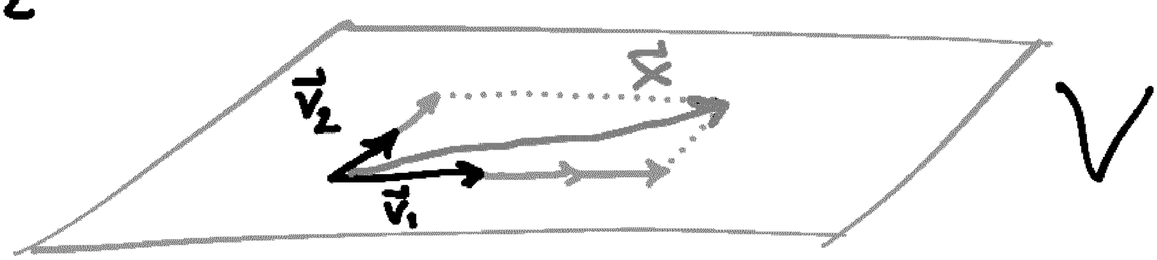
$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 = c_1\begin{bmatrix}1\\1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & 2\\1 & 3\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$$

We solve this system:

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 1 & 3 & 9 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right] \quad \text{Solution } \begin{cases} c_1 = 3 \\ c_2 = 2 \end{cases}.$$

So $\vec{x} = 3\vec{v}_1 + 2\vec{v}_2$. This means that $\vec{x}$ belongs to the plane spanned by $\vec{v}_1, \vec{v}_2$



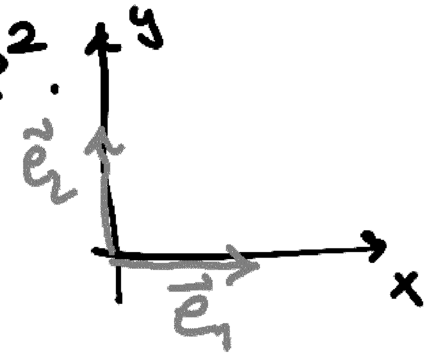
We imagine $\vec{v}_1$ and $\vec{v}_2$ form new coordinate axes inside the plane V. In this <u>new</u> coordinate system, the coordinates of $\vec{x}$ are $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

We let $B = (\vec{v}_1, \vec{v}_2)$ to mean the $\vec{v}_1$-$\vec{v}_2$-basis for V. Then we

Write $\boxed{[\vec{x}]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}}$

Coordinates of $\vec{x}$ in the basis B

<u>Ex</u>: Consider the plane $\mathbb{R}^2$.



$E = (\vec{e}_1, \vec{e}_2) = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$

E is the standard basis.

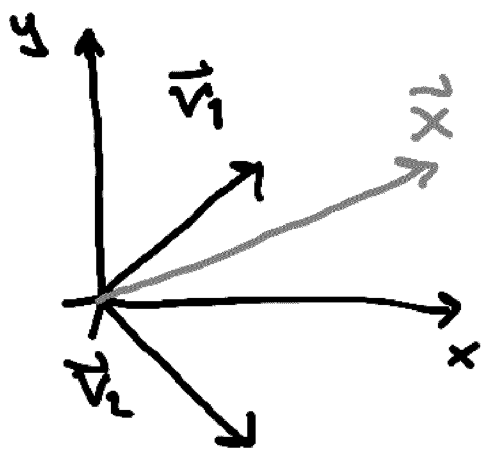
By default, when we describe a vector $\vec{x}$ in $\mathbb{R}^2$, we really describe it in terms of the standard basis

$\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2\vec{e}_1 + \vec{e}_2$.

Could write $[\vec{x}]_E = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ but we usually don't when it's the standard basis.

Let's now pick another basis

$B = (\vec{v}_1, \vec{v}_2) = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)$



solving $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$ we find

$c_1 = 3/2$, $c_2 = 1/2$.

So $[\vec{x}]_B = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}$.

<u>Def</u>: Consider a basis $B = (\vec{v}_1, \dots, \vec{v}_m)$ of a subspace V in $\mathbb{R}^n$. Since B is a basis, any vector $\vec{x}$ in V can be written as

$$\vec{x} = c_1\vec{v}_1 + \cdots + c_m\vec{v}_m \quad (c_1, \dots, c_m \text{ scalars})$$

in a unique way. These scalars are called the <u>B-coordinates</u> of $\vec{x}$.

$$\boxed{[\vec{x}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}}$$

Note that we can now relate the vector $\vec{x}$ with its B-coordinates:

$$\vec{x} = c_1\vec{v}_1 + \cdots c_m\vec{v}_m = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$$

$$= S[\vec{x}]_B, \quad S = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}.$$

<u>Ex</u>: Back to previous example

$\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $B = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)$.

We found $[\vec{x}]_B = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}$. We may indeed compute

$$S = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$S[\vec{x}]_B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 3/2 + 1/2 \\ 3/2 - 1/2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \vec{x}.$$

<u>Ex</u>: In the first example

$B = (\vec{v}_1, \vec{v}_2) = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right)$, $[\vec{x}]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

$$S = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$S[\vec{x}]_B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3+2 \\ 3+4 \\ 3+6 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} = \vec{x}.$$

<u>Theorem</u>: If B is a basis of a subspace V in $\mathbb{R}^n$. Then

(a) $[\vec{x}+\vec{y}]_B = [\vec{x}]_B + [\vec{y}]_B$ for all vectors $\vec{x}$ and $\vec{y}$ in V

(b) $[k\vec{x}]_B = k[\vec{x}]_B$ for all vectors $\vec{x}$ in V and all scalars.

<u>Ex</u>: Consider the basis

$$B = (\vec{v}_1, \vec{v}_2) = \left(\begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}-1\\3\end{bmatrix}\right) \text{ of } \mathbb{R}^2.$$

(1) Let $\vec{x} = \begin{bmatrix}10\\10\end{bmatrix}$ and let's find $[\vec{x}]_B$.

We need to find c_1, c_2 so that

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2.$$

$$\begin{bmatrix}10\\10\end{bmatrix} = c_1\begin{bmatrix}3\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}3 & -1\\1 & 3\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$$

$$\left[\begin{array}{cc|c}3 & -1 & 10\\1 & 3 & 10\end{array}\right]^{-1/3} \longrightarrow \left[\begin{array}{cc|c}3 & -1 & 10\\0 & \frac{10}{3} & \frac{20}{3}\end{array}\right]\cdot\frac{3}{10} \longrightarrow \left[\begin{array}{cc|c}3 & -1 & 10\\0 & 1 & 2\end{array}\right]_1$$

$$\longrightarrow \left[\begin{array}{cc|c}3 & 0 & 12\\0 & 1 & 2\end{array}\right]\cdot\frac{1}{3} \longrightarrow \left[\begin{array}{cc|c}1 & 0 & 4\\0 & 1 & 2\end{array}\right]$$

$c_1 = 4$, $c_2 = 2$. $\quad [\vec{x}]_B = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

(2). If $[\vec{y}]_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, let's find $\vec{y}$.

$S = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$, then

$$\vec{y} = S[\vec{y}]_B = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 6+1 \\ 2-3 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}.$$

If $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $T(\vec{x}) = A\vec{x}$ is a linear transformation, where A is an $(n \times n)$-matrix, then

if $B = (\vec{v}_1, \dots, \vec{v}_n)$ is a basis for $\mathbb{R}^n$, how is $[\vec{x}]_B$ related to $[T(\vec{x})]_B$?

By definition, if $S = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}$ then

$\vec{x} = S[\vec{x}]_B$. So

$$T(\vec{x}) = A\vec{x} = AS[\vec{x}]_B.$$

We also know

$$T(\vec{x}) = S[T(\vec{x})]_B$$

$$\Rightarrow \quad S\,[T(\vec{x})]_B = AS\,[\vec{x}]_B$$

The matrix S is actually always invertible, so

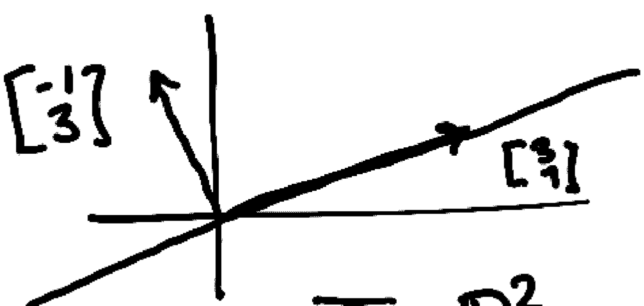
$$[T(\vec{x})]_B = (S^{-1}AS)\,[\vec{x}]_B.$$

Let $C = S^{-1}AS$, then

$$[T(\vec{x})]_B = C\,[\vec{x}]_B.$$

The matrix C is called the <u>B-matrix of T</u>.

<u>Ex</u>: Let $L = \text{span}\left(\begin{bmatrix}3\\1\end{bmatrix}\right)$ in $\mathbb{R}^2$



L (line w/ slope $1/3$)

$$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ T(\vec{x}) = \begin{bmatrix} 9/10 & 3/10 \\ 3/10 & 1/10 \end{bmatrix} \vec{x}.$$

This linear transformation is orthogonal projection onto L.

If we change bases to

$$\mathfrak{B} = \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right)$$

in L

perpendicular to L

the matrix will have a much simpler form.

$S = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$. $\quad [T(\vec{x})]_{\mathfrak{B}} = C[\vec{x}]_{\mathfrak{B}}$

$$C = S^{-1}AS.$$

Let's find S^{-1}:

$$\left[\begin{array}{cc|cc} 3 & -1 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \xrightarrow{-1/3} \left[\begin{array}{cc|cc} 3 & -1 & 1 & 0 \\ 0 & \frac{10}{3} & -\frac{1}{3} & 1 \end{array}\right] \cdot \frac{3}{10}$$

$$\rightarrow \left[\begin{array}{cc|cc} 3 & -1 & 1 & 0 \\ 0 & 1 & -\frac{1}{10} & \frac{3}{10} \end{array}\right] \xrightarrow{+1} \left[\begin{array}{cc|cc} 3 & 0 & \frac{9}{10} & \frac{3}{10} \\ 0 & 1 & -\frac{1}{10} & \frac{3}{10} \end{array}\right] \cdot \frac{1}{3}$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{10} & \frac{1}{10} \\ 0 & 1 & -\frac{1}{10} & \frac{3}{10} \end{array}\right] \quad S^{-1} = \frac{1}{10}\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}.$$

$$C = S^{-1}AS = \frac{1}{10}\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 9/10 & 3/10 \\ 3/10 & 1/10 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$$

$$= \cdots = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$[T(\vec{x})]_B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} [\vec{x}]_B.$$

Matrix of T with respect to the basis $\mathcal{B} = \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}\right)$ is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.