MAT 203 MIDTERM II

MONDAY APRIL 13, 2026
5:00–6:20PM

Name: ______________________________ ID: ______________

Instructions.

(1) Fill in your name and Stony Brook ID number.

(2) This exam is closed-book; no electronic devices. You are only allowed to have one (1) sheet of your own notes.

(3) You have 80 minutes to complete this exam.

(4) You must justify all your answers and show all your work. Even a correct answer without any justification will result in no credit.

1. (a) (3 pts) Show that the limit $\lim_{(x,y)\to(0,0)} \frac{x}{x+y}$ does not exist.

Solution. The limit along the line $x = 0$ is given by

$$\lim_{y\to 0} \frac{0}{0+y} = 0,$$

while the limit along the line $y = 0$ is given by

$$\lim_{x\to 0} \frac{x}{x} = 1,$$

and since these limits are different, the limit does not exist. □

(b) (5 pts) Show that the function $f(x, y) = 1-(x-1)^2-y^2$ is differentiable at $(x, y) = (0, 0)$.

Solution. We first compute the error term $E(x, y) = f(x, y) - f(0, 0) - f_x(0, 0)x - f_y(0, 0)y$. The partial derivatives of f are:

$$f_x(x, y) = -2(x - 1), \quad f_y(x, y) = -2y,$$

so $f_x(0, 0) = 2$ and $f_y(0, 0) = 0$, and therefore

$E(x, y) = (1 - (x - 1)^2 - y^2) - 0 - 2x = 1 - (x^2 - 2x + 1) - y^2 - 2x = -x^2 - y^2$,

so

$$\lim_{(x,y)\to(0,0)} \frac{E(x, y)}{\sqrt{x^2 + y^2}} = \lim_{(x,y)\to(0,0)} \frac{-x^2 - y^2}{\sqrt{x^2 + y^2}} = \lim_{(x,y)\to(0,0)} -\sqrt{x^2 + y^2} = 0,$$

showing that f is differentiable at $(x, y) = (0, 0)$. □

(c) (2 pts) Find the tangent plane to the graph of the function $f(x, y) = 1 - (x - 1)^2 - y^2$ at the point $(x, y) = (0, 0)$.

Solution. From part (b), we have $f_x(0, 0) = 2$ and $f_y(0, 0) = 0$. The equation for the tangent plane is therefore

$$f_x(0, 0)(x - 0) + f_y(0, 0)(y - 0) - (z - f(0, 0)) = 0$$
$$\iff 2x - z = 0$$

□

2. Consider the function $f(x, y) = x^2 - 2y$.

(a) (2 pts) Find all critical points for $f(x, y)$.

Solution. The critical points are solutions to the equation

$$\nabla f(x, y) = \langle 2x, -2 \rangle = \langle 0, 0 \rangle.$$

We see that there are no solutions to this equation, so there are no critical points of the function. □

(b) (8 pts) Find the global maximum, and global minimum of $f(x, y)$ in the region of the xy-plane bounded by the curve $x^2 + y^2 = 4$.

Solution. First, from part (a), we know that $f(x, y)$ has no critical points.
We need to find the critical points along the boundary of the region, that is, along the curve $x^2 + y^2 = 4$. To find these points, we use Lagrange multipliers. Setting $g(x, y) = x^2 + y^2 - 4$, the equation $x^2 + y^2 = 4$ is equivalent to $g(x, y) = 0$. We want to solve the system of equations

$$\begin{cases} \nabla f(x, y) = \lambda \nabla g(x, y) \\ g(x, y) = 0 \end{cases} \iff \begin{cases} 2x = 2\lambda x \\ -2 = 2\lambda y \\ x^2 + y^2 = 4 \end{cases}.$$

The first equation is equivalent to $2x(\lambda - 1) = 0$, and this can only be true if either $x = 0$ or $\lambda = 1$. In the first case we get $y = \pm 2$, and in the second case we get $y = -1$ which in the third equation gives $x^2 + 1 = 4$, and therefore $x = \pm\sqrt{3}$. This leads to the boundary critical points $(0, \pm 2)$ and $(\pm\sqrt{3}, -1)$. The values of f at these points are

$$f(0, 2) = -4,\ f(0, -2) = 4,\ f(\sqrt{3}, -1) = 5,\ f(-\sqrt{3}, -) = 5.$$

The maximum is therefore equal to 5 and the minimum is equal to -4. □

3. (10 pts) Compute the double integral $\iint_D \frac{2x}{1+y^2}\,dxdy$ where D is the region in $\{x \geq 0\}$ bounded by the curves $y = x^2$ and $y = 1$.

Solution. We will integrate in the x direction first, so

$$\iint_D \frac{2x}{1+y^2}\,dxdy = \int_0^1 \left(\int_0^{\sqrt{y}} \frac{2x}{1+y^2}\,dx \right) dy = \int_0^1 \left[\frac{x^2}{1+y^2} \right]_{x=0}^{x=\sqrt{y}} dy$$
$$= \int_0^1 \frac{y}{1+y^2}\,dy.$$

We compute this integral via the u-substitution $u = 1 + y^2$, which gives $\frac{1}{2}du = ydy$. Therefore

$$\int_0^1 \frac{y}{1+y^2}\,dy = \frac{1}{2}\int_1^2 \frac{1}{u}\,du = \frac{1}{2}\left[\ln(u)\right]_1^2 = \frac{1}{2}\ln(2).$$

□

4. (10 pts) Compute the integral $\iiint_E (x^2+y^2)\,dxdydz$ where E is the region bounded by the paraboloid $z = 1 - x^2 - y^2$, and the plane $z = 0$.

Solution. We first integrate in the z-direction, and afterwards we are left with a double integral over the region D which is the region bounded by $x^2 + y^2 = 1$.

$$\iiint_E x^2{+}y^2\,dxdydz = \iint_D x^2{+}y^2 \left(\int_0^{1-x^2-y^2} dz\right) dxdy = \iint_D (x^2{+}y^2)(1{-}x^2{-}y^2)\,dxdy.$$

We change to polar coordinates, and get

$$\iint_D (x^2+y^2)(1-x^2-y^2)\,dxdy = \int_0^{2\pi}\int_0^1 r^2(1-r^2)\,rdrd\theta = 2\pi\int_0^1 r^3(1-r^2)\,dr$$
$$= 2\pi\left[\frac{r^4}{4} - \frac{r^6}{6}\right]_0^1 = \frac{\pi}{6}.$$

□