MAT 203

PRACTICE PROBLEMS

1. Find a vector $\vec{v}$ that has magnitude 7, and points in the same direction as the vector $\vec{u} = \langle 3, 4\rangle$.
2. Consider the two vectors $\vec{a} = \langle 2, -4\rangle$ and $\vec{b} = \langle -1, 2\rangle$. Find two scalars α and β such that $\alpha\vec{a} + \beta\vec{b} = \vec{0}$.
3. Let t be some scalar, and consider the two vectors $\vec{u} = \langle 2\cos t, -2\sin t, 3\rangle$ and $\vec{v} = \langle 0, 0, 3\rangle$. Find the magnitude of the vectors $\vec{u} - \vec{v}$, and the vector $-2\vec{u}$.
4. Let $P = (x, y, z)$ be be a point in space that is an equal distance away from the point $A = (1, -1, 0)$, and $B = (-1, 2, 1)$. Show that the point P lies on the plane satisfying the equation $-2x + 3y + z = 2$.
5. Determine the scalar α such that the vectors $\vec{a} = \langle 2, 3\rangle$ and $\vec{b} = \langle 9, \alpha\rangle$ are orthogonal.
6. Find all vectors in space that are orthogonal to the vector $\vec{v} = \langle 1, -1, -1\rangle$.
7. Find a vector $\vec{w}$ of magnitude 3 such that $\vec{w} \times \langle 1, 0, -1\rangle = \langle 3, 0, 3\rangle$.
8. Find a vector of magnitude 10 that is orthogonal to the plane passing through the x-axis and the point $P = (1, 2, 4)$.
9. Consider the following two parametrizations of two lines $\vec{r}(t) = \langle -2 + 2t, -6, 2 + 6t\rangle$ and $\vec{s}(t) = \langle -1 + t, 1 + t, t\rangle$. Are the lines perpendicular to each other?
10. Consider the planes given by the equations $-y + z - 2 = 0$ and $x - y = 0$.
(a) Show that the planes intersect in a line.
(b) Find a parametrization of the line passing through the point $P = (-8, 0, 2)$ that is parallel to the line of intersection of the two planes.
11. Identify the quadratic surface $5x^2 - 4y^2 + 20z^2 = 0$.
12. Identify the quadratic surface $x^2 + z^2 - 4y + 4 = 0$.
13. Evaluate the limit $\lim_{t\to 0} \langle e^t, \frac{\sin t}{t}, e^{-t}\rangle$.
14. Parametrize the curve in the plane described by the equation $4x^2 + 9y^2 = 36$.
15. Let $\vec{u}(t) = \langle t^2, -2t, 1\rangle$. Compute $\frac{d}{dt}(\vec{u}(t) \times \vec{u}'(t))$.
16. Find the arc length of the curve parametrized by $\vec{r}(t) = \langle -t, 4t, 3t\rangle$ for $0 \leq t \leq 1$.
17. Compute the limit $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{(x-y)}$ or prove that it does not exist.
18. Compute the limit $\lim_{(x,y,z)\to(1,2,3)} \frac{xz^2-y^2z}{xyz-1}$ or prove that it does not exist.
19. Compute the limit $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ or prove that it does not exist.
20. Is the function

$$f(x, y) = \begin{cases} \frac{x^2y}{x^2+y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$$

continuous at $(x, y) = (0, 0)$?
21. Let $f(x, y) = \arctan\left(\frac{y}{x}\right)$. Find $f_x(2, -2)$ and $f_y(2, -2)$.
22. Given $f(x, y, z) = e^{-2x}\sin(z^2y)$, show that $f_{xyy} = f_{yxy}$.
23. Show that $f(x, t) = e^{-t}\cos\left(\frac{x}{c}\right)$, where c is a scalar, satisfies the so-called *heat equation* $\frac{\partial f}{\partial t} = c^2\frac{\partial^2 f}{\partial x^2}$.
24. Find the tangent plane of the surface described by $z = \sin x + \sin y + \sin(x + y)$ at the point $(0, 0, 0)$.
25. Find the tangent plane of the surface described by $x^3 + y^3 = 3xyz$ at the point $\left(1, 2, \frac{3}{2}\right)$.
26. Prove that the function $f(x, y) = xe^y$ is differentiable at $(x, y) = (0, 0)$.
27. Find the directional derivative of $f(x, y) = \ln(5x + 4y)$ at $(x, y) = (3, 9)$ in the direction of $\vec{u} = \langle 6, 8\rangle$.
28. Find the unit vector along which the function $f(x, y) = \arctan\left(\frac{y}{x}\right)$ increases most rapidly at $(x, y) = (-9, 9)$.
29. Find and classify all critical points of the function $f(x, y) = e^{-x^2-y^2-2x}$.
30. Find and classify all critical points of the function $f(x, y) = x^3 - xy + y^2 - 1$.
31. Find the points on the surface $x^2 - yz = 5$ that are closest to the origin.

32. Find the maximum volume of a rectangular box with three faces in the coordinate planes and a vertex in the first octant on the plane $x + y + z = 1$.

33. Find the global maximum and minimum values of the function $f(x, y) = x^2 + y^2 - 2y + 1$ in the region $R = \{(x, y) \mid x^2 + y^2 \leq 4\}$.

34. Maximize the function $f(x, y) = 2x + 3y + 5z$ on the sphere $x^2 + y^2 + z^2 = 19$.

35. Find the global maximum and minimum values of the function $f(x, y) = 3x^3 - y^2$ in the region of the xy-plane bounded by the graphs of the functions $y = x^2$, and $y = \sqrt{x}$.

36. Compute the double integral $\iint_D x\,dxdy$ where D is the region in the plane bounded by the graph of $f(x) = \sin x$, and the x-axis for $0 \leq x \leq \pi$.

37. Compute the double integral $\iint_R e^x\,dxdy$ where R is the region in the plane bounded by the x-axis, and the two lines $y = x$ and $y = 3 - 2x$.

38. Compute the double integral $\iint_R xy\,dxdy$ where R is the set of all points (x, y) in the plane satisfying $y \leq \frac{x^2}{2}$ and $x^2 + y^2 \leq 3$.

39. Compute the double integral $\iint_D \frac{e^{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}\,dxdy$ where D is the region in the third quadrant in the plane that lies between the two circles centered at the origin of radii 1 and 2.

40. Compute the triple integral $\iiint_P 2z\,dxdydz$ where P is the solid pyramid in the first octant bounded by the plane $z = 10 - 2x - y$.

41. Compute the triple integral $\iiint_S dxdydz$ where S is the region in space bounded below by the plane $z = 1$, and bounded above by the sphere $x^2 + y^2 + z^2 = 4$.

42. Compute the triple integral $\iiint_B \frac{z\cos(\sqrt{x^2+y^2+z^2})}{\sqrt{x^2+y^2+z^2}}\,dxdydz$, where B is the region in the first octant bounded by the two spheres centered at the origin of radii 1 and 2.

43. Determine whether the vector field $\vec{F}(x, y, z) = \langle ye^z, xe^z, xye^z\rangle$ is conservative and, if so, find a potential function.

44. Determine whether the vector field $\vec{F}(x, y) = \langle e^x\cos y, 6e^x\sin y\rangle$ is conservative and, if so, find a potential function.

45. Consider the vector field $\vec{F}(x, y) = \langle x^2y^2, xy\rangle$ and evaluate the line integral $\int_C \vec{F}\cdot d\vec{r}$, where C is the curve in the plane consisting of the y-axis for $0 \leq y \leq 1$, the x-axis for $0 \leq x \leq 1$, and the part of the unit circle in the first quadrant, oriented counterclockwise.

46. Consider the vector field $\vec{F}(x, y) = \langle 2xy, x\rangle$, and compute the line integral $\int_C \vec{F}\cdot d\vec{r}$, where C is the ellipse $x^2 + 2x + 4y^2 = 0$, oriented counterclockwise.

47. For any differentiable vector field F in space, prove $\nabla\cdot(\nabla\times\vec{F}) = 0$.

48. For any differentiable scalar function f, prove $\nabla\times(\nabla f) = \vec{0}$.

49. Compute the surface integral of the vector field $\vec{F}(x, y, z) = \langle 1, 1, z\rangle$ of the unit sphere centered at $(0, 0, 0)$.

50. Let $\vec{F}(x, y, z) = \langle x + y + z, y, 2x - y\rangle$ and consider the surface S consisting of the cylinder $x^2 + y^2 = 1$ for $0 \leq z \leq 3$, including the top and the bottom of the cylinder. We assume S is oriented with the outwards normal. Compute the surface integral $\iint_S \vec{F}\cdot d\vec{r}$.