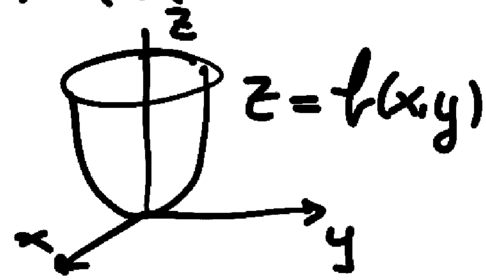
MAT 203 L17 Wed Apr 15

§6.1 Vector fields

Definition: A vector field is an assignment of a vector at every point in the plane, or in space.

Ex: Gradient vector field of a function $f(x,y)$.

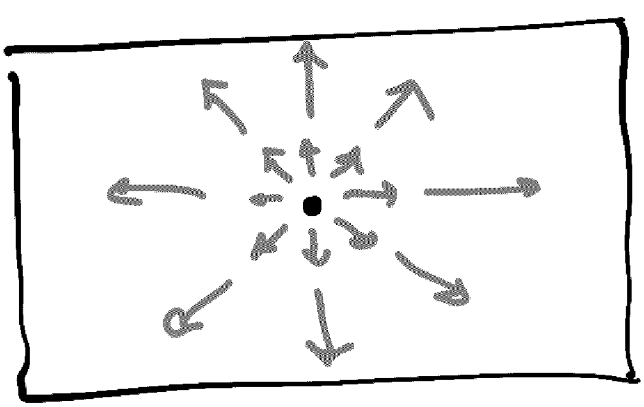


e.g. $f(x,y) = x^2+y^2$

$\nabla f(x,y) = \langle 2x, 2y \rangle$.

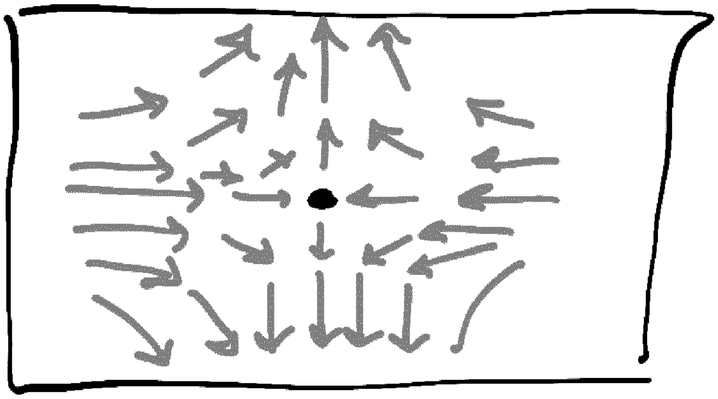
So in the plane we draw the vector $\langle 2x,2y\rangle$ at position (x,y):

xy-plane



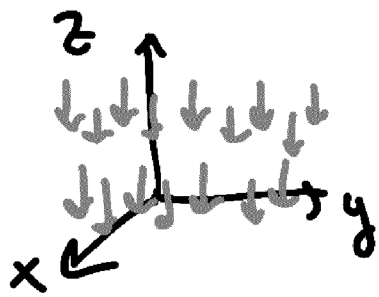
e.g. $f(x,y) = x^2 - y^2$

$\nabla f(x,y) = \langle 2x, -2y \rangle$



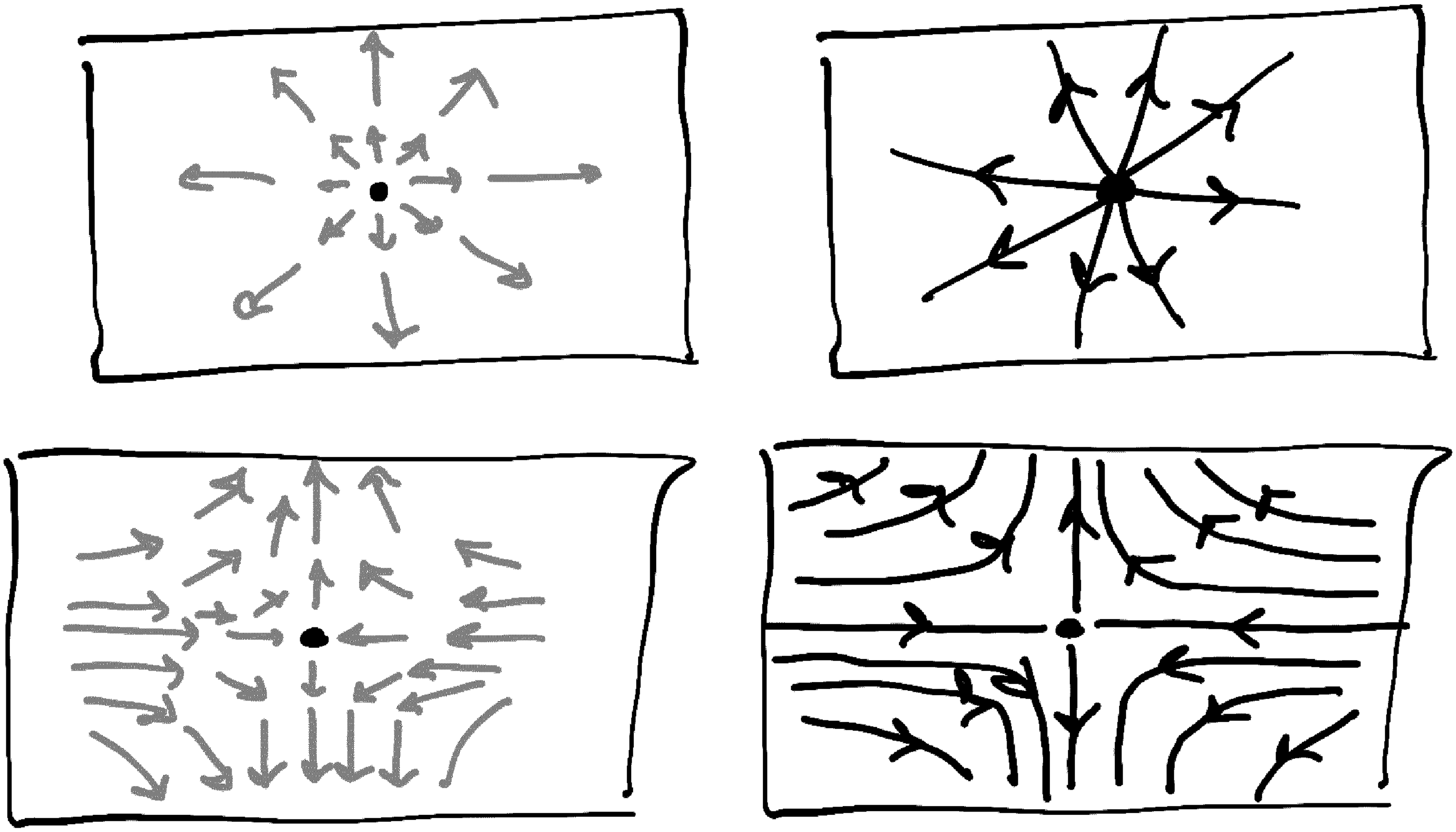
xy-plane

<u>Ex</u>: Gravitational vector field:
It's a vector field in space that assigns every pt to the vector
$\langle 0, 0, -g \rangle$ ($g \approx 9.82\ m/s^2$)



<u>Definition</u>: A flow line of a vector field is a curve that is tangent to the vector field.

Ex. Vector field $\qquad$ flow lines



Conservative vector fields:

A vector field $\vec{F}$ is called "conservative" if

$\boxed{\vec{F} = \nabla f}$ for some function f.

If $\vec{F}$ is conservative, such a function is called a <u>potential function</u>.

We will later see why this is important.

Ex: $f(x,y,z) = x^2yz - \sin(xy)$ is a potential function for

$$\vec{F}(x,y,z) = \langle 2xyz - y\cos(xy), x^2z - x\cos(xy), x^2y \rangle$$

Since we may compute ∇f, and we get

$$\nabla f = \langle 2xyz - y\cos(xy), x^2z - x\cos(xy), x^2y \rangle$$
$$= \vec{F}.$$

Theorem: If f and g are potential functions for a vector field $\vec{F}$, then $f = g + \text{Constant}$.

Ex. Find a potential function for $\vec{F}(x,y,z) = \langle 0, 0, -g \rangle$

By inspection we see that
$f(x,y,z) = -gz$ is a potential.
So every potential is $-gz + C$, where
C is a constant.

Ex. Find a potential for
$\vec{F}(x,y) = \langle y, x+\cos(y) \rangle$.

Sol: If there's a potential
$f(x,y)$, then

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle y, x+\cos(y) \rangle$$

$$\begin{cases} \dfrac{\partial f}{\partial x} = y \\[2ex] \dfrac{\partial f}{\partial y} = x+\cos(y) \end{cases}$$

we now solve this system.

1^{st} eq gives $f(x,y) = xy + g(y)$

then $\dfrac{\partial f}{\partial y} = x + g'(y) = x + \cos(y)$

$\iff g'(y) = \cos(y) \iff g(y) = \sin(y) + C$

So $f(x,y) = xy + \sin(y) + C$ is a potential.

__Ex__, Find a potential for $\vec{F}(x,y,z) = \langle 7, -2, x^3 \rangle$.

__Sol__: Like above we try to solve $\nabla f = \vec{F}$.

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle 7, -2, x^3 \rangle$$

$$\begin{cases} \partial f/\partial x = 7 \\ \partial f/\partial y = -2 \\ \partial f/\partial z = x^3 \end{cases}$$

1^{st} eq $\Rightarrow f(x,y,z) = 7x + g(y,z)$.

Inputting it into 2^{nd} eq gives

$$\frac{\partial g}{\partial y} = -2 \Rightarrow g(y,z) = -2y + h(z)$$

Now inserting it into 3rd equ

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(7x-2y+h(z)\right) = \boxed{h'(z) = x^3}$$

Since left hand side only depends on z, and right hand side only depends on x, they can not be equal for all (x,y,z).

Conclusion: There is no potential!