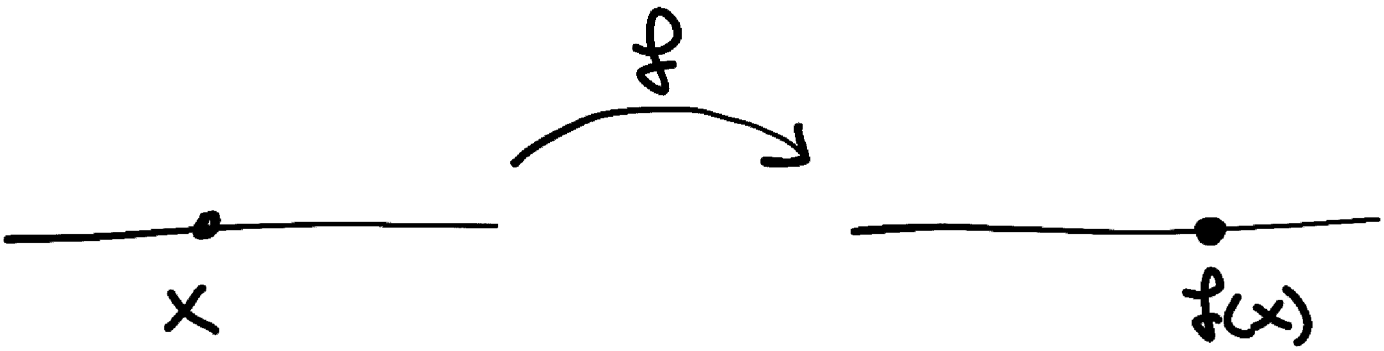
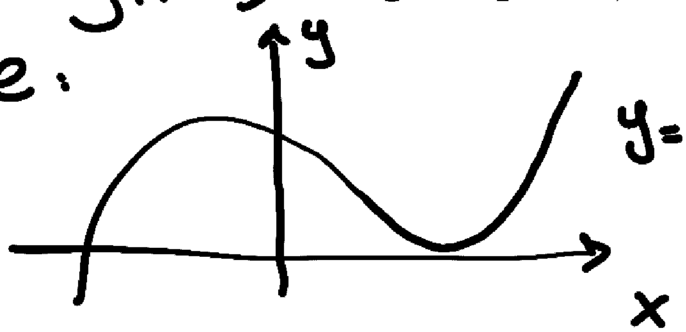
MAT 203 L9 Tue Sep 30

§4.1 Functions of Several variables

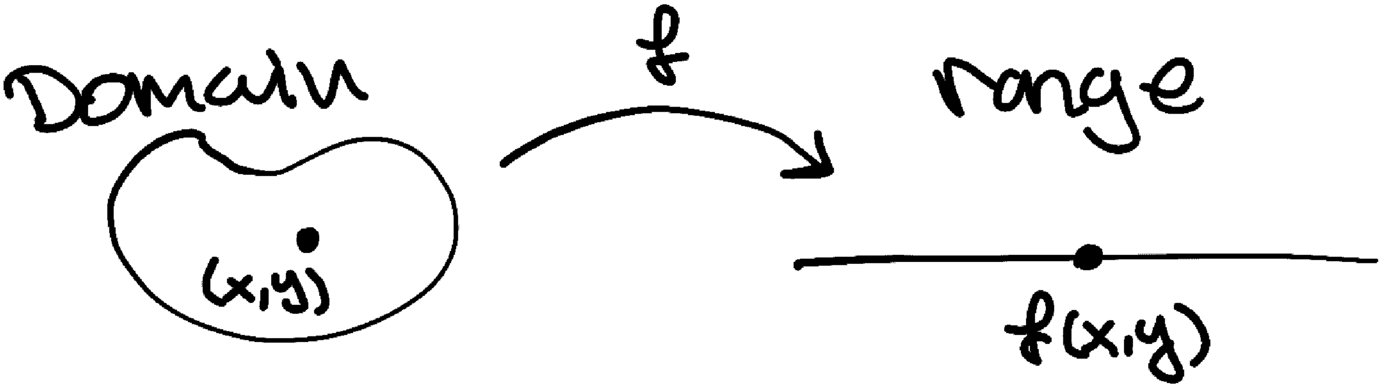
Recall from Calc I/II that a function $f(x)$ (of 1 variable) has 1 input and 1 output; typically both input and outputs are real numbers (scalars):



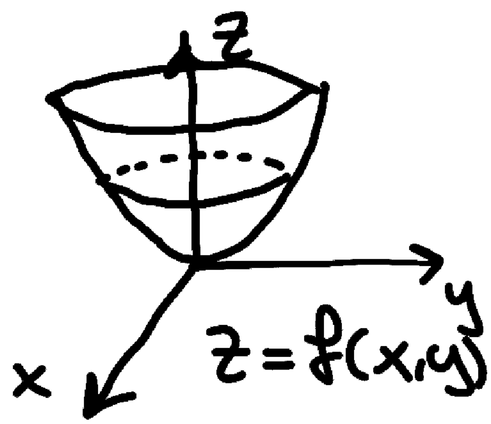
<u>Graph</u>: Plotting all points (x,y) in the plane such that $y=f(x)$ often gives a curve in the plane.



Now: We will allow 2 inputs $f(x,y)$. Inputs and output are still real numbers (scalars)



Graph: The graph will be a surface in 3D



EX: $f(x,y) = x^2 + y^2$

graph is an elliptic paraboloid.

Def: A function of two variables maps each ordered pair (x,y) in a subset D of the plane $\mathbb{R}^2$ to a unique real number z.

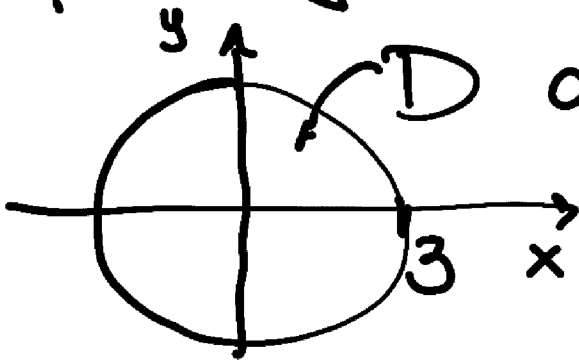
D = "domain"

"range" = set of all numbers z such that $z = f(x,y)$, for some (x,y) in D

Typically, the domain for us will be the "largest possible."

Ex The domain of $f(x,y) = \sqrt{9-x^2-y^2}$ is all (x,y) such that $9-x^2-y^2 \geq 0$ since we do not allow complex numbers as outputs. Domain is

$$9 - x^2 - y^2 \geq 0 \iff x^2 + y^2 \leq 9$$



D all pts in disk with radius 3.

Range is $[0,3]$ because

$$f(x,y) = \sqrt{9-x^2-y^2} \geq 0, \text{ and}$$

$$9 - x^2 - y^2 \leq 9, \text{ so } \sqrt{9-x^2-y^2} \leq \sqrt{9} = 3$$

Ex: $f(x,y) = 3x+5y+2$.

Domain is all (x,y) in $\mathbb{R}^2$, since the expression is defined for all x,y.

Range is all real numbers:

Given any z, we can always find x,y such that

$$\boxed{z = 3x+5y+2}$$

Of course we can also consider functions that depend on 3 variables (or more...), too: $f(x,y,z)$, $f(x,y,z,w)$, etc.

Ex: $f(x,y,z) = y\sin(x) + z^3xy$

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§4.2 Limits and Continuity

Recall limits from Calc I again:

If L is a real number s.th

$L = \lim_{x \to a} f(x)$, it means that "when x is very close to a, $f(x)$ will be very close to L"

A little more precisely, the distance $|f(x) - L|$ can be made as small as we wish, by making the distance $|x - a|$ very very small.

<u>Def</u>: Let L be a real number. We say $L = \lim_{(x,y) \to (a,b)} f(x,y)$ if

$|f(x,y) - L|$ can be made arbitrarily small by making the distance between (x,y) and (a,b) small.

"As (x,y) gets close to (a,b), $f(x,y)$ will get close to L."

<u>Ex</u>: $\lim_{(x,y) \to (1,1)} x + 2y = 3$

because when (x,y) is very close to $(1,1)$, the value of $f(x,y) = x+2y$ is very close to 3.

We now need some limit laws to help us to compute limits.

<u>Theorem:</u>

Let $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$, $c =$ constant

$\lim\limits_{(x,y)\to(a,b)} g(x,y) = M$

(1) $\lim\limits_{(x,y)\to(a,b)} c = c$

(2) $\lim\limits_{(x,y)\to(a,b)} x = a$, $\lim\limits_{(x,y)\to(a,b)} y = b$

(3) $\lim\limits_{(x,y)\to(a,b)} (f(x,y) \pm g(x,y)) = L \pm M$

(4) $\lim\limits_{(x,y)\to(a,b)} c f(x,y) = cL$

(5) $\lim\limits_{(x,y)\to(a,b)} f(x,y) g(x,y) = LM$

(6) $\lim\limits_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$ if $M \neq 0$

(7) $\lim\limits_{(x,y)\to(a,b)} \left(f(x,y)\right)^n = L^n$, $n \geq 1$ integer

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<u>Ex</u>: (1) $\lim\limits_{(x,y)\to(1,1)} (x+y)\cdot 3x$

$= \left(\lim\limits_{(x,y)\to(1,1)} x+y\right)\left(\lim\limits_{(x,y)\to(1,1)} 3x\right)$

$= \left[\left(\lim\limits_{(x,y)\to(1,1)} x\right)+\left(\lim\limits_{(x,y)\to(1,1)} y\right)\right]\left(\lim\limits_{(x,y)\to(1,1)} 3x\right)$

$= [1 + 1](3) = 6.$

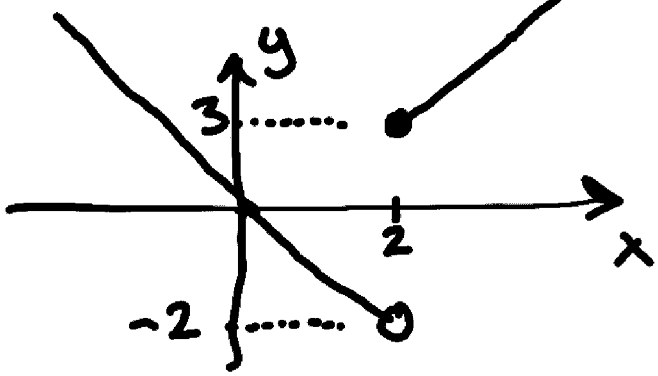
(2) $\lim\limits_{(x,y)\to(2,-1)} \dfrac{2x+3y}{4x-3y} = \dfrac{\lim\limits_{(x,y)\to(2,-1)} 2x+3y}{\lim\limits_{(x,y)\to(2,-1)} 4x-3y}$

$= \dfrac{2\cdot 2+3(-1)}{4\cdot 2-3(-1)} = \dfrac{1}{11}$

To understand how limits can

fail to exist, let's consider the case w/ 1 variable:

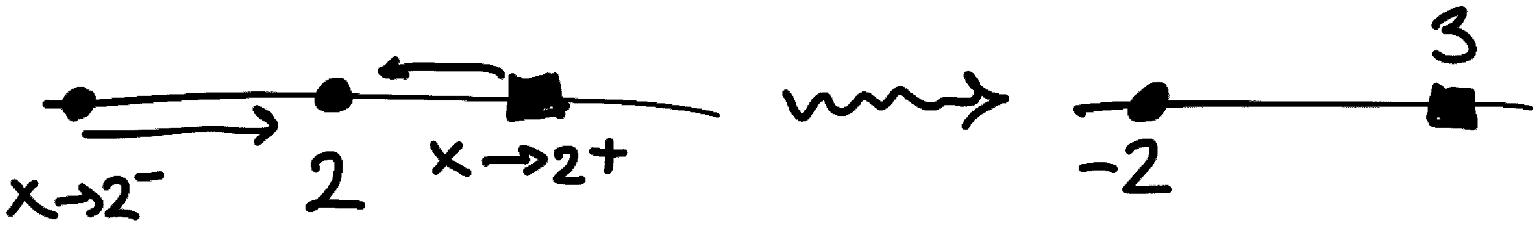
<u>Ex</u>: $f(x) = \begin{cases} x+1, & x \geq 2 \\ -x & x < 2 \end{cases}$



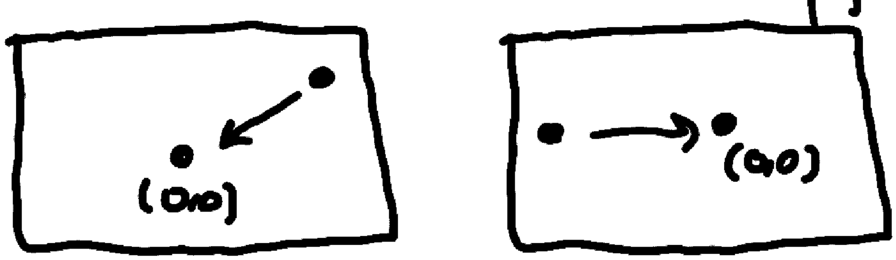
$\lim_{x \to 2^+} f(x) = 3$

$\lim_{x \to 2^-} f(x) = -2$

When we approach $x=2$ from the left & from the right, we get <u>different resuts</u>.



For limits where $(x,y) \to (0,0)$, there are infinitely many directions along which (x,y) can approach $(0,0)$:

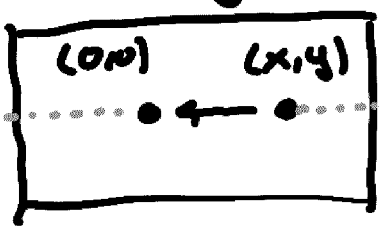


Limits that fail to exist:

Ex (1) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$.

$$f(x,y) = \frac{xy}{x^2+y^2}$$

If $(x,y) \longrightarrow (0,0)$ along the x-axis

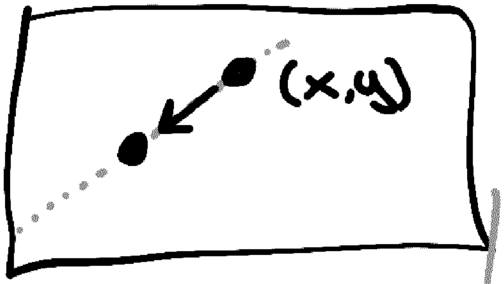


, meaning if $y=0$ and $x \to 0$,

$$f(x,0) = \frac{x \cdot 0}{x^2+0^2} = 0,$$ then

$$\boxed{\lim_{x\to 0} f(x,0) = 0.}$$

If $(x,y) \to (0,0)$ along the line $x=y$ in the plane,



$$f(x,x) = \frac{x^2}{x^2+x^2} = \frac{1}{2}$$

$$\boxed{\lim_{x\to 0} f(x,x) = \frac{1}{2}.}$$

therefore, if $(x,y) \to (0,0)$ along x-axis, and along the line $x=y$, we get two different resuts, so $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ doesn't exist.

(2) $\lim\limits_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+3y^4}$

$f(x,y) = \frac{4xy^2}{x^2+3y^4}$. If $x=y$,

$f(x,x) = \frac{4x^3}{x^2+3x^4} = \frac{4x}{1+3x^2}$

$\longrightarrow \frac{0}{1+0} = 0$ as $x \to 0$

If $x=y^2$, $f(y^2,y) = \frac{4y^4}{y^4+3y^4}$

$= \frac{4y^4}{4y^4} = 1 \longrightarrow 1$ as $y \to 0$

so $\lim\limits_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+3y^4}$ doesn't exist.