MAT 203 L8 Thu Sep 18

Recall: $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

- Arc length between $a \le t \le b$:

$$L = \int_a^b \|\vec{r}\,'(t)\|\, dt$$

- Curvature at time t

$$\kappa = \frac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|} = \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}.$$

- Principal unit normal vector:

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|}$$

- Binormal vector: $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$

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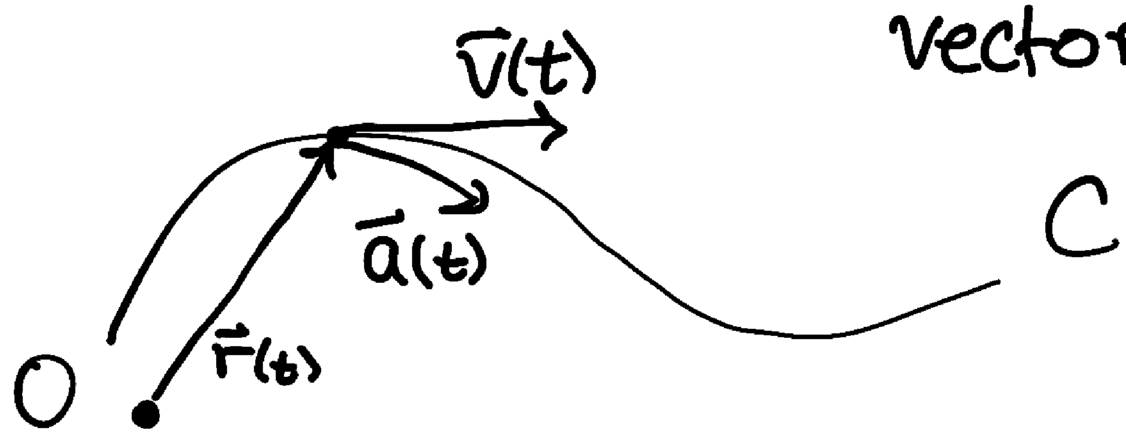
§3.4 Motion in Space

If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, then
$\vec{r}(t)$ = position at time t.

$\vec{v}(t) = \vec{r}\,'(t) =$ Velocity Vector

$\|\vec{v}(t)\| = \|\vec{r}\,'(t)\| =$ Speed

$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t) =$ acceleration vector.



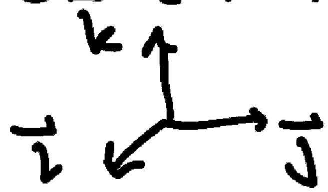
When describing the motion of objects in space, its the most convenient to describe it with respect to a "moving" frame of reference.

The standard unit vectors:

$\vec{i} = \langle 1,0,0 \rangle$, $\vec{j} = \langle 0,1,0 \rangle$, $\vec{k} = \langle 0,0,1 \rangle$

is a fixed (non-moving) frame

$(\vec{i}, \vec{j}, \vec{k})$

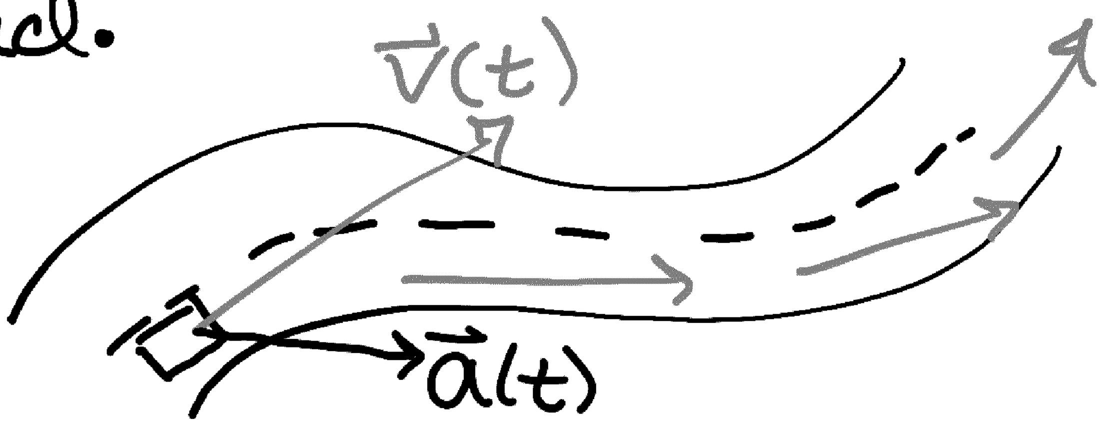


An example of a moving frame is the "Frenet frame":

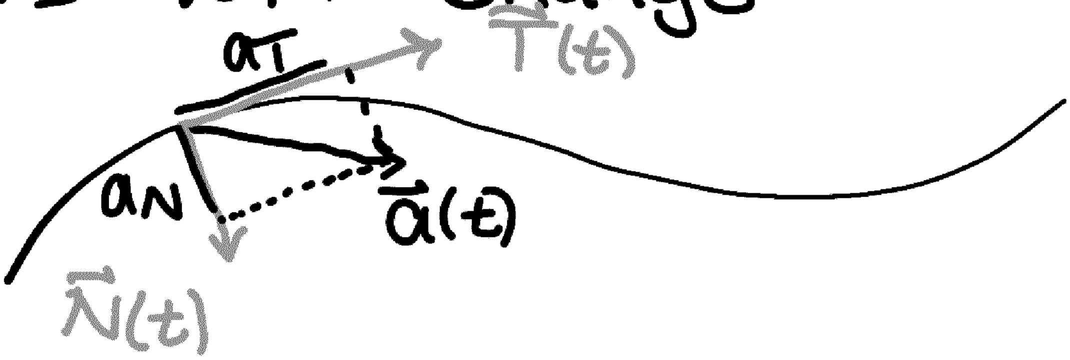
$$(\vec{T}(t), \vec{N}(t), \vec{B}(t))$$

(or "TNB-frame")

<u>Ex</u>: Suppose you drive with constant <u>speed</u> along a curvy road.



Even though speed is constant, both the velocity & acceleration vectors will change.



The acceleration can be written as

$$\vec{a}(t) = a_T \vec{T}(t) + a_N \vec{N}(t)$$

a_T = tangential acceleration
a_N = normal acceleration.

Since $\vec{T}(t)$ and $\vec{N}(t)$ are unit vectors, a_T and a_N are easy to compute:

$$a_T(t) = \vec{a}(t) \cdot \vec{T}(t) = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{\|\vec{r}'(t)\|}$$

$$a_N(t) = \vec{a}(t) \cdot \vec{N}(t) = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|}$$

<u>EX</u>: Let $\vec{r}(t) = \langle t^2, 2t-3, 3t^2-3 \rangle$ and let's compute $a_T(t)$ and $a_N(t)$.

$$\vec{r}\,'(t) = \langle 2t, 2, 6t \rangle$$

$$\vec{r}\,''(t) = \langle 2, 0, 6 \rangle$$

$$a_T(t) = \frac{\langle 2t, 2, 6t \rangle \cdot \langle 2, 0, 6 \rangle}{\sqrt{(2t)^2 + 2^2 + (6t)^2}}$$

$$= \frac{4t + 36t}{\sqrt{4t^2 + 4 + 36t^2}} = \frac{40t}{\sqrt{40t^2+4}} = \frac{20t}{\sqrt{10t^2+1}}$$

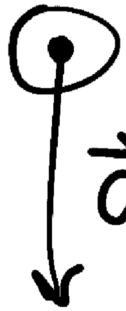
$$\vec{r}\,'(t) \times \vec{r}\,'(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t & 2 & 6t \\ 2 & 0 & 6 \end{vmatrix} = \langle 12, 0, -4 \rangle$$

$$a_N(t) = \frac{\| \langle 12, 0, -4 \rangle \|}{\| \vec{r}\,'(t) \|} = \frac{\sqrt{144+16}}{\sqrt{40t^2+4}}$$

$$= \sqrt{\frac{160}{40t^2+4}} = \sqrt{\frac{40}{10t^2+1}}$$

Projectiles:

If we drop an object (and ignore any air resistance), it will start falling downwards with constant acceleration $g = 9.82\, m/s^2$



(x_0, y_0, z_0)

$\vec{a}(t) = \langle 0, 0, -g \rangle$

By integration we can find the Velocity & position :

$\vec{a}(t) = \vec{v}\,'(t)$, so $\vec{v}(t) = \int a(t)\,dt$

$= \int \langle 0,0,-g \rangle\, dt = \langle 0,0,-g \rangle t + \langle c_1, c_2, c_3 \rangle$

$\langle C_1, C_2, C_2 \rangle =$ Velocity at time $t = 0 = \vec{v}(0)$.

$\vec{v}(t) = \vec{v}(0) + \langle 0,0,-g \rangle t.$

Next, $\vec{r}\,'(t) = \vec{v}(t)$, so

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \int \vec{v}(0) + \langle 0,0,-g \rangle t\, dt$$

$$= \langle D_1, D_2, D_3 \rangle + \vec{v}(0)t + \frac{\langle 0,0,-g \rangle t^2}{2}$$

where $\langle D_1, D_2, D_3 \rangle$ = position at time $t=0 = \vec{r}(0) = \langle x_0, y_0, z_0 \rangle$

$$\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + \vec{v}(0)t + \frac{\langle 0,0,-g \rangle t^2}{2}$$

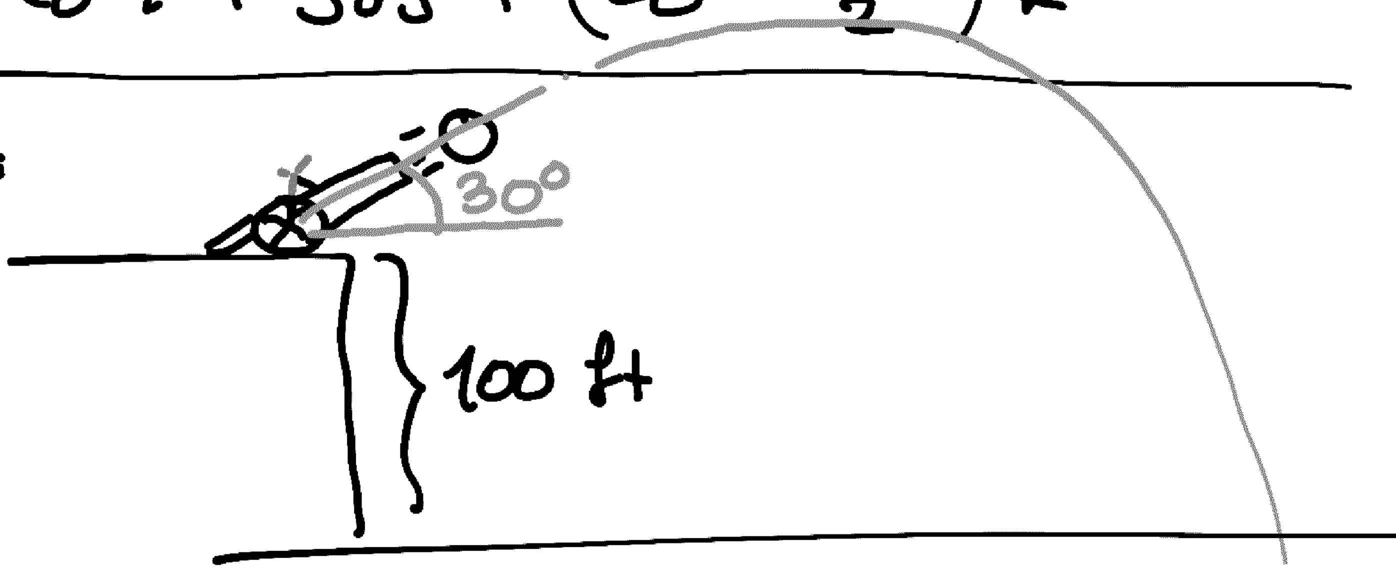
If we simply drop the object with no initial velocity, then the motion is simply described by:

$$\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + \frac{\langle 0,0,-g \rangle t^2}{2}$$

$$= \left\langle x_0, y_0, z_0 - \frac{gt^2}{2} \right\rangle$$

$$= x_0 \vec{i} + y_0 \vec{j} + \left(z_0 - \frac{gt^2}{2} \right) \vec{k}$$

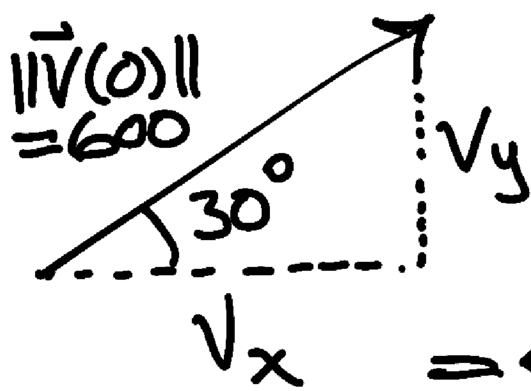
Ex:



Suppose a cannonball is shot out of a cannon with initial speed $v = 600$ ft/s at an angle of $30°$. Let's find the max height & position when it hits the ground below.

Sol: Initial position $\vec{r}(0) = \langle 0, 100 \rangle$

Initial Velocity: $30° = \frac{\pi}{6}$ rad



$\|\vec{v}(0)\| = 600$

$\vec{v}(0) = \langle v_x, v_y \rangle$

$= \langle 600 \cos \frac{\pi}{6}, 600 \sin \frac{\pi}{6} \rangle$

$= \langle \frac{600 \cdot \sqrt{3}}{2}, \frac{600}{2} \rangle = \langle 300\sqrt{3}, 300 \rangle$

Acceleration: $\vec{a}(t) = \langle 0, -g \rangle$.

We found above that

$$\vec{r}(t) = \langle x_0, y_0 \rangle + \vec{v}(0)t + \frac{\vec{a}(t)t^2}{2}$$

$$= \langle 0, 100 \rangle + \langle 300\sqrt{3}, 300 \rangle t + \frac{\langle 0, -g \rangle t^2}{2}$$

$$= \left\langle 300\sqrt{3}t, 100 + 300t - \frac{gt^2}{2} \right\rangle$$

To find max height, we need to maximize

$$100 + 300t - \frac{gt^2}{2} = f(t).$$

Since $g \approx 32\ ft/s^2$,

$$f(t) = -16t^2 + 300t + 100$$

$$= -16\left(t^2 - \frac{300}{16}t - \frac{100}{16}\right)$$

$$= -16\left(\left(t - \frac{300}{32}\right)^2 - \left(\frac{300}{32}\right)^2 - \frac{100}{16}\right)$$

$$= -16\left(t - \frac{300}{32}\right)^2 + 16\left(\frac{300}{32}\right)^2 + 100$$

Max height $= 16\left(\frac{300}{32}\right)^2 + 100$ ft

≈ 1506 ft (above ground level)

It hits the ground when $y=0$, so

$$-16t^2+300t+100=0$$

Which gives $t \approx 19.078$ s.

The position at this time is

$$\vec{r}(19.078) \approx \langle 300\sqrt{3}\cdot 19.078, 0\rangle$$

$$\approx \langle 9913, 0\rangle$$

So it hits the ground 9913 ft away from the cliff.