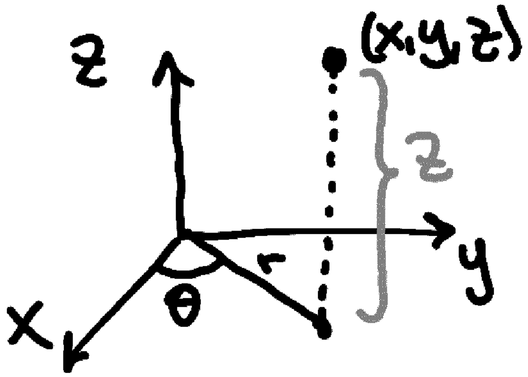
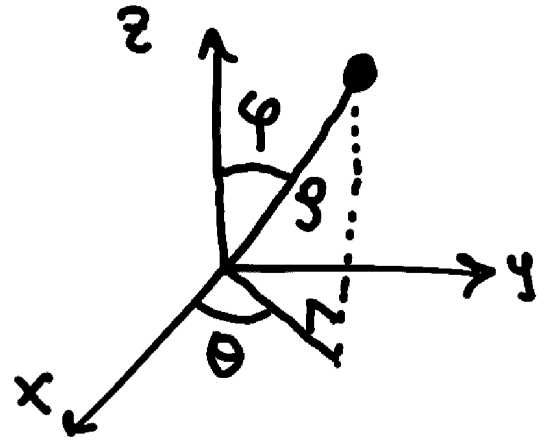
MAT 203 L7 Tue Sep 16

Recall: • Cylindrical coords (r, θ, z)



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

• Spherical coords (ρ, θ, φ)



$$\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases}$$

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

• $\lim_{t\to a} \vec{r}(t) = \langle \lim_{t\to a} x(t), \lim_{t\to a} y(t), \lim_{t\to a} z(t) \rangle$

• $\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$

Later, it will be convenient to normalize the tangent vector $\vec{r}'(t)$ of a parametrized curve.

Define
$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}, \text{ provided}$$
that $\|\vec{r}'(t)\| \neq 0$.
It is called the "principal tangent vector."

Integrals

Let $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$. If $x(t)$, $y(t)$, and $z(t)$ are continuous functions over $[a,b]$, then
$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \right\rangle$$

Also, this is of course true for indefinite integrals, too:

$$\int \vec{r}(t)\,dt = \left\langle \int x(t)\,dt, \int y(t)\,dt, \int z(t)\,dt \right\rangle$$

Ex: $\vec{r}(t) = \langle 3t^2+2t, 3t-6, 8t^3 \rangle$

$$\int \vec{r}(t)\,dt = \left\langle \int 3t^2+2t, \int 3t-6\,dt, \int 8t^3\,dt \right\rangle$$

$$= \left\langle t^3+t^2+C_1, \frac{3t^2}{2}-6t+C_2, 4t^4+C_3 \right\rangle$$

Note: We need 3 different constant (one in each component).

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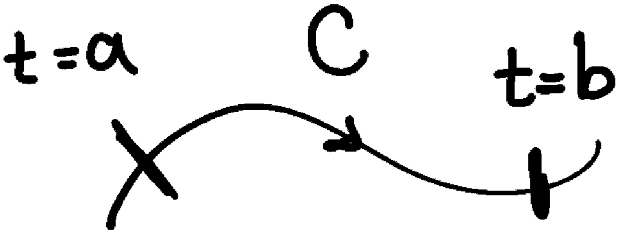
§3.3 Arc length and Curvature

Recall from lecture 4 that we derived the following formula for the arc length of a curve C with parametrization

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$:



$$A = \int_a^b \|\vec{r}\,'(t)\|\, dt$$
$$= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$$

<u>Ex</u>: If $y = f(x)$, then $\begin{cases} x = t \\ y = f(t) \end{cases}$ gives a parametrization of its graph:

$\vec{r}(t) = \langle t, f(t) \rangle$. Then $\vec{r}\,'(t) = \langle 1, f'(t) \rangle$, so the arc length between $t=a$ and $t=b$ is

$$\int_a^b \|\vec{r}\,'(t)\|\, dt = \int_a^b \sqrt{1 + (f'(t))^2}\, dt.$$

Ex: Find the arc length of $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ (helix) for: $0 \le t \le 2\pi$.

$\vec{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$

$\|\vec{r}'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1}$

$= \sqrt{1+1} = \sqrt{2}$

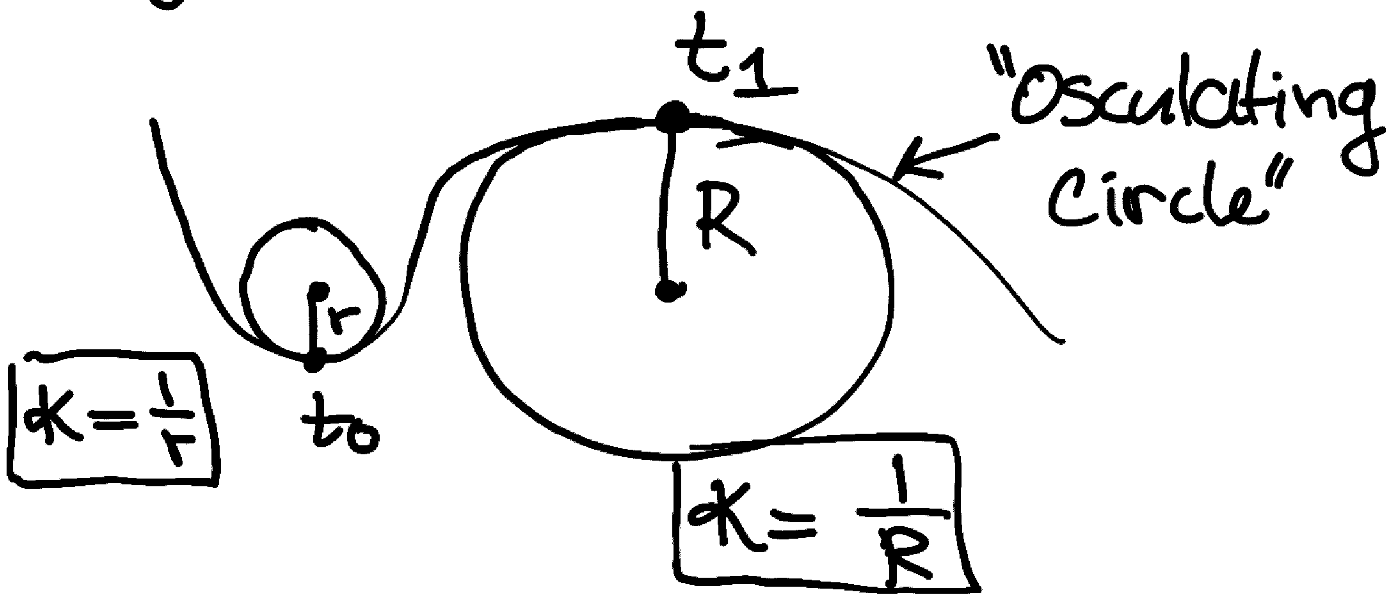
$$\int_0^{2\pi} \|\vec{r}'(t)\|\, dt = \int_0^{2\pi} \sqrt{2}\, dt = \sqrt{2}\Big[t + C\Big]_0^{2\pi}$$

$= 2\sqrt{2}\pi.$

Definition If C is a smooth curve, its curvature at time t is

$$\boxed{\kappa = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}}$$

The curvature measures how sharply the curve "turns."



The radius of the so-called "osculating circle" is the reciprocal of the curvature.

<u>Ex</u>: Compute the curvature of $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$.

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{\langle -\sin t, \cos t, 1 \rangle}{\sqrt{2}}$$

$$\vec{T}\,'(t) = \left\langle -\frac{\cos t}{\sqrt{2}}, -\frac{\sin t}{\sqrt{2}}, 0 \right\rangle$$

$$\kappa = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \frac{\sqrt{\frac{\cos^2 t}{2} + \frac{\sin^2 t}{2}}}{\sqrt{2}}$$

$$= \frac{\sqrt{1/2}}{\sqrt{2}} = \frac{1}{2}$$

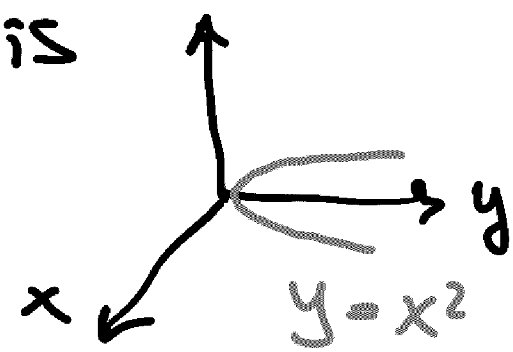
The curvature can also be computed as

$$\kappa = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}.$$

This is assuming that the curve belongs to space.
(Since the cross product only makes sense in space!)

If $y = f(x)$, a parametrization of its graph in space is

$$\vec{r}(t) = \langle t, f(t), 0 \rangle$$



It is contained

in the xy-plane, meaning $z=0$.

Now, let's compute the curvature:

$$\vec{r}\,'(t) = \langle 1, f'(t), 0\rangle$$

$$\vec{r}\,''(t) = \langle 0, f''(t), 0\rangle$$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & f'(t) & 0 \\ 0 & f''(t) & 0 \end{vmatrix}$$

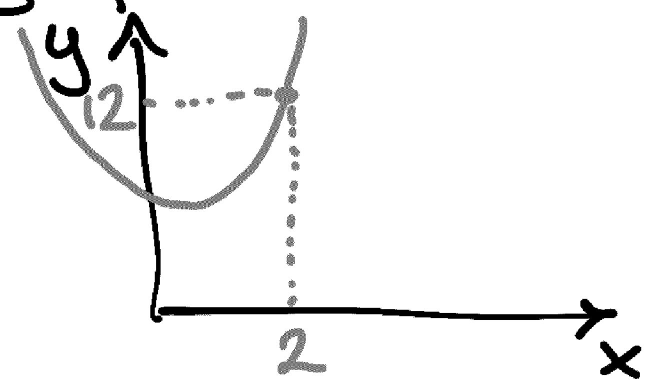
$$= \underbrace{\begin{vmatrix} f'(t) & 0 \\ f''(t) & 0 \end{vmatrix}}_{=0} \vec{i} - \underbrace{\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}}_{=0} \vec{j} + \begin{vmatrix} 1 & f'(t) \\ 0 & f''(t) \end{vmatrix} \vec{k}$$

$$= f''(t)\vec{k} = \langle 0, 0, f''(t)\rangle.$$

$$\kappa = \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3} = \frac{\|\langle 0,0,f''(t)\rangle\|}{\|\langle 1, f'(t), 0\rangle\|^3}$$

$$= \frac{|f''(t)|}{(1+(f'(t))^2)^{3/2}}.$$

Ex: Let's find the curvature of the graph of $f(x) = 3x^2 - 2x + 4$ at $x = 2$



$f'(x) = 6x - 2$, $f''(x) = 6$. Therefore, at $x = 2$ we have

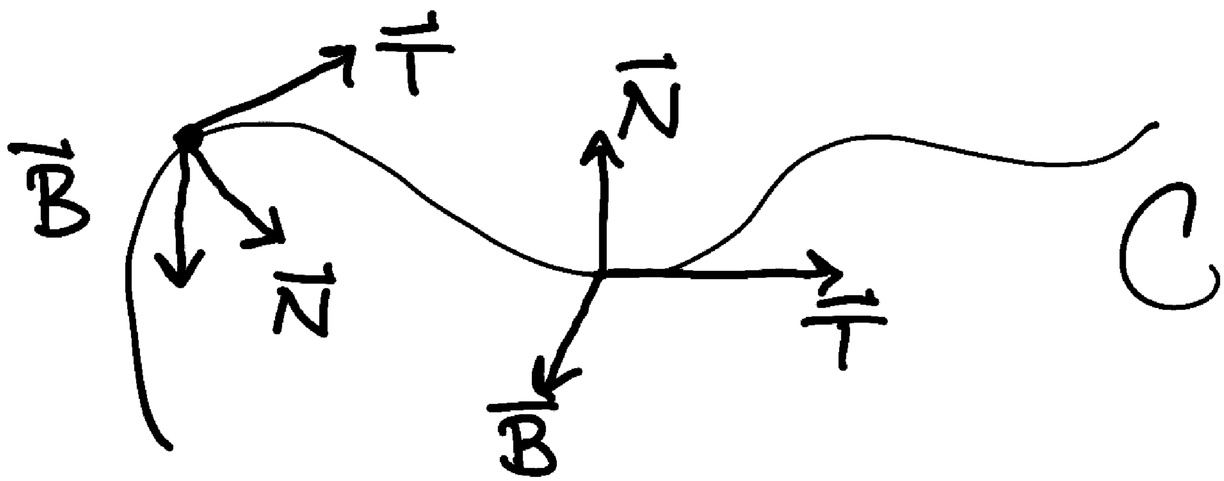
$$\kappa = \frac{|f''(2)|}{(1 + (f'(2))^2)^{3/2}} = \frac{6}{(1+10^2)^{3/2}} = \frac{6}{(101)^{3/2}}$$

$$= \frac{6}{101\sqrt{101}} \approx 0.006$$

Definition: If C is a smooth curve in space, and $\vec{T}'(t) \neq 0$, then the "principal unit normal vector" is

$$\boxed{\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}.}$$

Definition: The "binormal vector" is $\boxed{\vec{B}(t) = \vec{T}(t) \times \vec{N}(t).}$



$\vec{r}(t)$ = position

$\vec{T}(t)$ = normalized velocity

$\vec{N}(t)$ = normalized acceleration
