

Recall: Quadratic surfaces:

Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



Hyperboloids

one-sheeted: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$



two-sheeted: $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$



Cone: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$



Paraboloid:

elliptic $\frac{x^2}{a^2} + \frac{y^2}{b^2} = z$



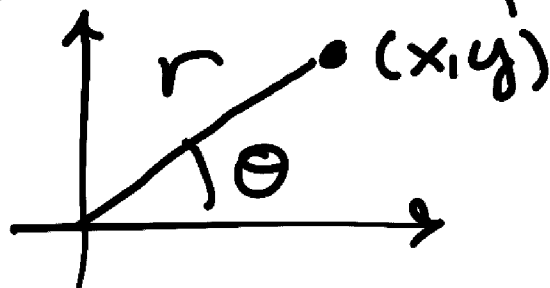
hyperbolic $\frac{x^2}{a^2} - \frac{y^2}{b^2} = z$



§2.7 Cylindrical and Spherical coordinates

To parametrize conic sections, it's usually convenient to use polar coords (r, θ) in the plane

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$

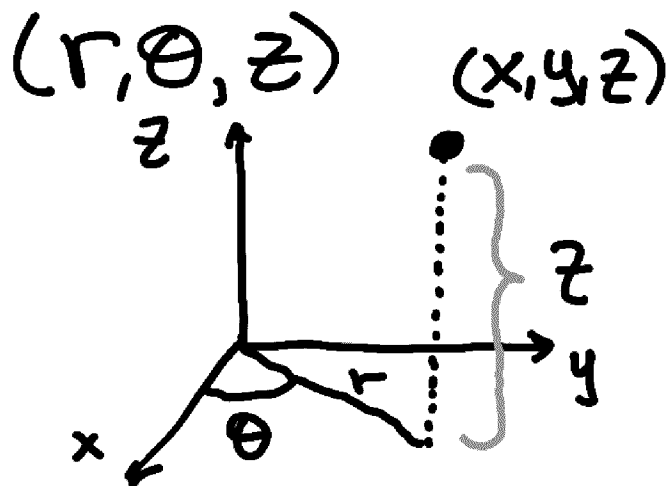


In space, there are two convenient coordinate systems.

Cylindrical coordinates:

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases}$$

$$r \geq 0, 0 \leq \theta \leq 2\pi$$



Can solve for r, θ, z in terms of x, y, z :

$$\begin{cases} r^2 = x^2 + y^2 \\ \tan \theta = y/x \\ z = z \end{cases}$$

Ex: Paraboloid $x^2 + y^2 = z$

In cylindrical coordinates:

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases} \quad \text{then } z = x^2 + y^2 = r^2$$

$$\vec{r}(r, \theta) = \langle r \cos \theta, r \sin \theta, r^2 \rangle$$

Spherical coords:

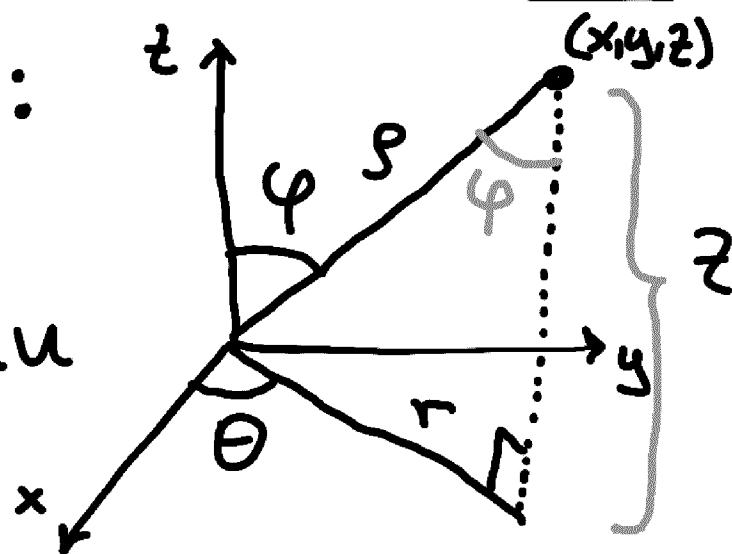
We derive the
Spherical coords in
two steps:

1. Cylindrical coords

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases}$$

2. Now

$$\begin{cases} r = \rho \sin \varphi \\ z = \rho \cos \varphi \end{cases}$$



Which gives spherical coords:

$$\begin{cases} x = \rho \sin \varphi \cos \theta \\ y = \rho \sin \varphi \sin \theta \\ z = \rho \cos \varphi \end{cases} \quad (\rho, \theta, \varphi)$$

$$\rho \geq 0, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \varphi \leq \pi$$

Can also solve for x, y, z :

$$\begin{cases} \rho^2 = x^2 + y^2 + z^2 \\ \tan \theta = y/x \\ \varphi = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) \end{cases}$$

Ex: Ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{then}$$

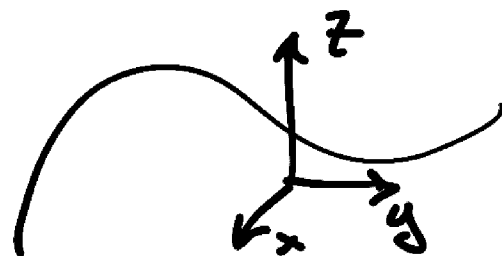
$$\begin{cases} x = a \sin \varphi \cos \theta \\ y = b \sin \varphi \sin \theta \\ z = c \cos \varphi \end{cases}$$

$$\vec{r}(\theta, \varphi) = \langle a \sin \varphi \cos \theta, b \sin \varphi \cos \theta, c \cos \varphi \rangle$$

§3.1 Vector-valued functions

We have already discussed parametrizations of curves in space:

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$



this is a vector-valued function
input is a scalar, but the output is a vector.

$$\langle x(t), y(t), z(t) \rangle = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

We will now discuss limits and continuity.

Def: A vector-valued function $\vec{r}(t)$ approaches the limit $\vec{L}$ as $t \rightarrow a$

if $\lim_{x \rightarrow a} \|\vec{\gamma}(t) - \vec{L}\| = 0.$

In this case we write

$$\lim_{x \rightarrow a} \vec{\gamma}(t) = \vec{L}.$$

Note that $\|\vec{\gamma}(t) - \vec{L}\|$ is a function w/ scalars as input and scalars as output. So the limit $\lim_{x \rightarrow a} \|\vec{\gamma}(t) - \vec{L}\|$ is exactly the kind we studied in Calculus I!

In practice, to compute limits we will use the following:

Theorem: If $\vec{\gamma}(t) = \langle x(t), y(t), z(t) \rangle$, then

$$\lim_{t \rightarrow a} \vec{\gamma}(t) = \left\langle \lim_{t \rightarrow a} x(t), \lim_{t \rightarrow a} y(t), \lim_{t \rightarrow a} z(t) \right\rangle$$

provided that the limits

$\lim_{t \rightarrow a} x(t)$, $\lim_{t \rightarrow a} y(t)$, and $\lim_{t \rightarrow a} z(t)$ exist. (If one of them doesn't, the limit $\lim_{t \rightarrow a} \vec{\gamma}(t)$ doesn't exist!)

Ex: ①

$$\lim_{t \rightarrow 3} \langle t^2 - 3t + 4, 4t + 3 \rangle = \langle 9 - 9 + 4, 12 + 3 \rangle \\ = \langle 4, 15 \rangle$$

$$\textcircled{2} \lim_{t \rightarrow 3} \left\langle \frac{2t-4}{t+1}, \frac{t}{t^2+1}, 4t-3 \right\rangle \\ = \left\langle \frac{1}{2}, \frac{3}{10}, 9 \right\rangle$$

Remember that $f(x)$ is continuous at $x=a$ if $\lim_{x \rightarrow a} f(x)$ exists and is equal to $f(a)$.

Def: $\vec{\gamma}(t) = \langle x(t), y(t), z(t) \rangle$ is continuous at $t=t_0$ if

- 1) $\vec{\gamma}(t_0)$ exists

$$2) \lim_{t \rightarrow t_0} \vec{\gamma}(t) \text{ exists}$$

$$3) \lim_{t \rightarrow t_0} \vec{\gamma}(t) = \vec{\gamma}(t_0)$$

§3.2 Calculus of Vector-valued functions

We have already computed derivatives of parametrizations before to find tangents.

Let us articulate this "officially" now:

Def: The derivative of $\vec{\gamma}(t)$ is given by the limit

$$\lim_{h \rightarrow 0} \frac{\vec{\gamma}(t+h) - \vec{\gamma}(t)}{h} = \vec{\gamma}'(t)$$

If it exists, $\vec{\gamma}(t)$ is differentiable at t .

Thm: Assume $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$,
is differentiable at $t = t_0$. Then
 $\vec{r}'(t_0) = \langle x'(t_0), y'(t_0), z'(t_0) \rangle$

Ex: 1) $\vec{r}(t) = \langle 3 \cos t, 2 \sin t \rangle$

$$\vec{r}'(t) = \langle -3 \sin t, 2 \cos t \rangle$$

2) $\vec{r}(t) = \langle e^t \sin t, e^t \cos t \rangle$

$$\vec{r}'(t) = \langle e^t \sin t + e^t \cos t, e^t \cos t - e^t \sin t \rangle$$

3) $\vec{r}(\theta) = \langle \sin \theta \cos \theta, \sin \theta \sin \theta \rangle$

$$\vec{r}'(\theta) = \langle \cos^2 \theta - \sin^2 \theta, \cos \theta \sin \theta + \sin \theta \cos \theta \rangle$$

$$= \langle \cos(2\theta), \sin(2\theta) \rangle.$$

Properties of the derivative:

(i) $\frac{d}{dt}(c \vec{r}(t)) = c \vec{r}'(t)$

$$(ii) \frac{d}{dt}(\vec{r}(t) \pm \vec{u}(t)) = \vec{r}'(t) \pm \vec{u}'(t)$$

$$(iii) \frac{d}{dt}(f(t) \vec{u}(t)) = f'(t) \vec{u}(t) + f(t) \vec{u}'(t)$$

$$(iv) \frac{d}{dt}(\vec{r}(t) \cdot \vec{u}(t)) = \vec{r}'(t) \cdot \vec{u}(t) + \vec{r}(t) \cdot \vec{u}'(t)$$

$$(v) \frac{d}{dt}(\vec{r}(t) \times \vec{u}(t)) = \vec{r}'(t) \times \vec{u}(t) + \vec{r}(t) \times \vec{u}'(t)$$

$$(vi) \frac{d}{dt}(\vec{r}(f(t))) = \vec{r}'(f(t)) \cdot f'(t)$$

$$(vii) \text{ If } \vec{r}(t) \cdot \vec{r}(t) = c \text{ (a scalar)} \\ \text{then } \vec{r}(t) \cdot \vec{r}'(t) = 0.$$
