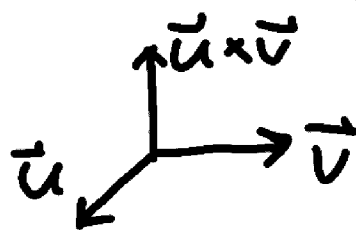


Recall: • Dot product.

$$\vec{u} = \langle u_1, u_2, u_3 \rangle, \vec{v} = \langle v_1, v_2, v_3 \rangle$$

$$\boxed{\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3}$$

• Cross product.



$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \vec{k}$$

$$= (u_2 v_3 - v_2 u_3) \vec{i} - (u_1 v_3 - v_1 u_3) \vec{j} \\ + (u_1 v_2 - v_1 u_2) \vec{k}$$

§2.5 Lines and planes in space

Recall that a non-vertical line in the

Plane has the eqn $y=kx+m$
Where $k = \text{Slope}$, $m = y\text{-intercept}$.

A vertical line has the eqn
 $x=a$.

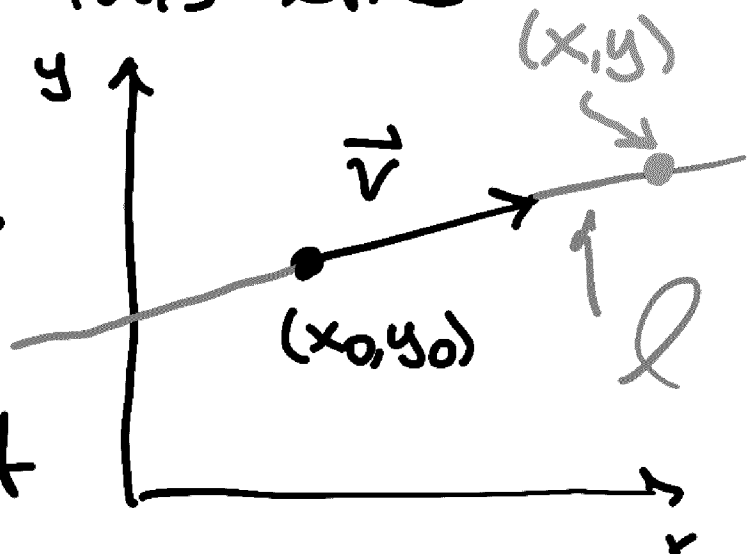
A more general way of writing
it: $ax + by + c = 0$ ($a, b \neq (0, 0)$).

If $b \neq 0$: $y = \underbrace{-\frac{a}{b}}_{=k} x - \underbrace{\frac{c}{b}}_{=m}$ (non-vertical line)

If $b = 0$: $x = -\frac{c}{a}$ (vertical line)

Will now describe this line
using vectors.

We now describe
the line passing
through the point



(x_0, y_0) and is parallel to the vector $\vec{v} = \langle a, b \rangle$. If (x, y) is any point on the line l , then

$\langle x - x_0, y - y_0 \rangle$ is parallel to $\vec{v}$, so it's a scalar multiple of $\vec{v}$:
So there is some scalar t such that $\langle x - x_0, y - y_0 \rangle = t\vec{v} = t\langle a, b \rangle$

$$\begin{cases} x - x_0 = ta \\ y - y_0 = tb \end{cases} \Rightarrow \begin{cases} x = x_0 + ta \\ y = y_0 + tb \end{cases}$$

or:

$$\boxed{\vec{r}(t) = \langle x_0, y_0 \rangle + t\langle a, b \rangle}$$

This is the parametric form of the line. The scalar t is called the parameter.

This also works in the same way in space:

The line passing through (x_0, y_0, z_0) that is parallel to $\vec{v} = \langle a, b, c \rangle$ is given by

$$\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle.$$

EX: Find the parametric eqn of the line passing through the points $(1, -3, 2)$ and $(5, -2, 8)$.

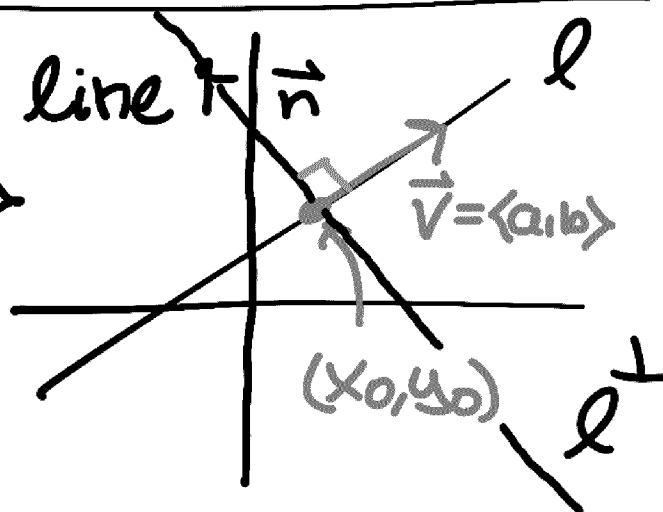
Sol: Let $\vec{v}$ be the vector from $(1, -3, 2)$ to $(5, -2, 8)$.

$$\vec{v} = \langle 5-1, -2-(-3), 8-2 \rangle = \langle 4, 1, 6 \rangle.$$

So the line is described by

$$\vec{r}(t) = \langle 1, -3, 2 \rangle + t \langle 4, 1, 6 \rangle$$

Now consider the line l parallel to $\vec{v} = \langle a, b \rangle$ that passes through (x_0, y_0) .



Let us find the line $\ell^\perp$ through (x_0, y_0) that is perpendicular to ℓ .

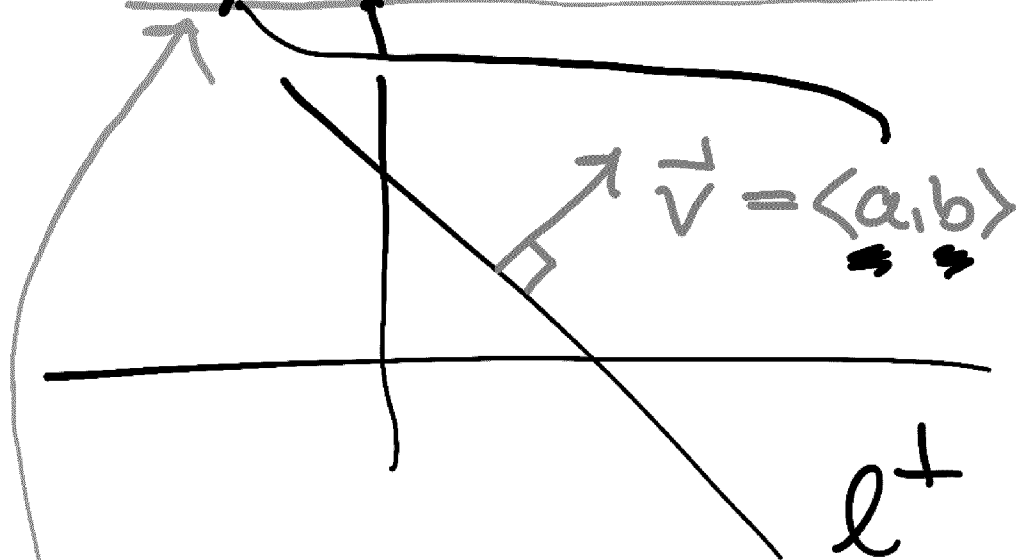
The direction vector $\vec{n}$ is orthogonal to $\vec{v}$, so $\vec{n} \cdot \vec{v} = 0$

If (x, y) is a point on $\ell^\perp$, then

$\vec{n} = \langle x - x_0, y - y_0 \rangle$, $\vec{v} = \langle a, b \rangle$ so

$$\langle x - x_0, y - y_0 \rangle \cdot \langle a, b \rangle = a(x - x_0) + b(y - y_0) = 0$$

$$\Leftrightarrow \boxed{ax + by - ax_0 - by_0 = 0}$$



equation for the line $\ell^\perp$

with normal vector $\langle a, b \rangle$ passing through (x_0, y_0) .

Q: How to Change between a parametric & linear form (meaning $ax+by+c=0$) of a line?

EX: Consider the line $2x+3y+6=0$
Find a parametric form for it.

Sol: $y = -\frac{2}{3}x - 2$.

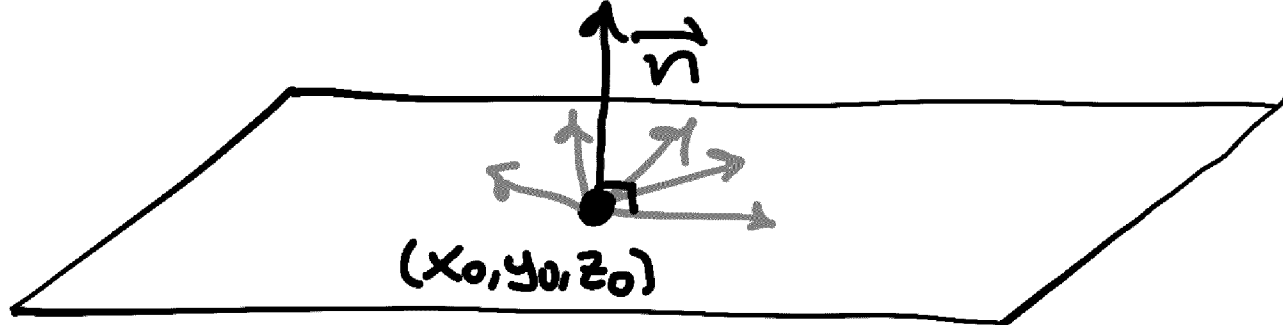
$(0, -2)$, $(3, -4)$ are two pts on the line, so a direction vector is $\vec{v} = \langle 3-0, -4-(-2) \rangle = \langle 3, -2 \rangle$.

Therefore a parametric form is given by

$$\boxed{\vec{r}(t) = \langle 0, -2 \rangle + t \langle 3, -2 \rangle}$$

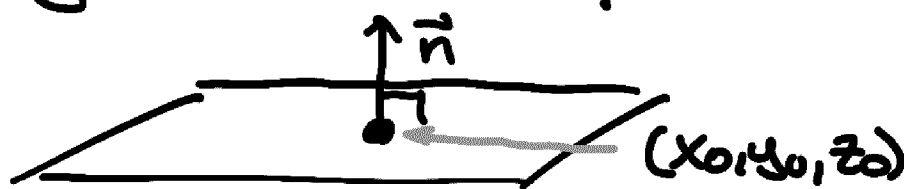
Lines are best described by a direction vector.

A plane has many directions
Since it's 2-dimensional:

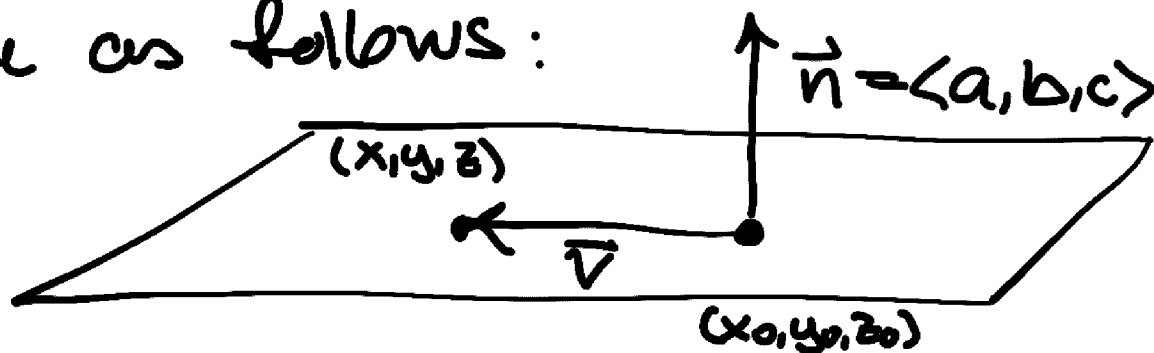


All of these vectors point along the plane.

The best way to describe a plane is by a normal vector: a vector that is orthogonal to any vector pointing along the plane.



If $\vec{n} = \langle a, b, c \rangle$ and the plane passes through (x_0, y_0, z_0) , then we can describe any other point (x, y, z) in the plane as follows:



$$\vec{v} = \langle x - x_0, y - y_0, z - z_0 \rangle \leftarrow \text{in the plane}$$

$$\vec{n} = \langle a, b, c \rangle \leftarrow \text{normal to the plane}$$

$$\begin{aligned}\vec{n} \cdot \vec{v} &= 0 \Leftrightarrow \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle \\ &= a(x - x_0) + b(y - y_0) + c(z - z_0) \\ &= ax + by + cz - ax_0 - by_0 - cz_0 = 0\end{aligned}$$

Linear form of a plane:

$$\boxed{ax + by + cz + d = 0} \quad (a, b, c) \neq (0, 0, 0)$$

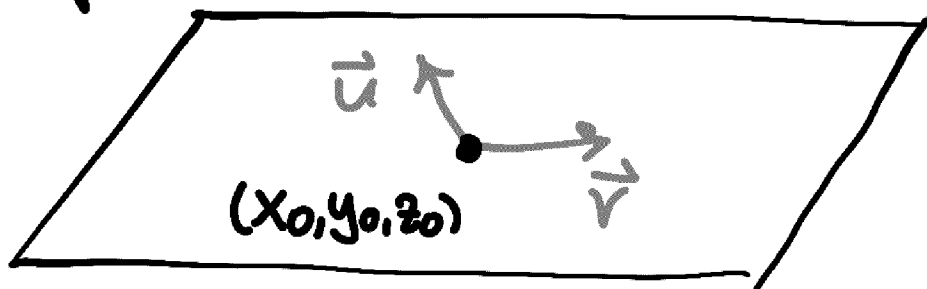
Normal to the plane:

$$\boxed{\vec{n} = \langle a, b, c \rangle.}$$

Parametric form of a plane

As mentioned above, a plane doesn't have one direction, since it is 2-dimensional.

In fact if $\vec{u}, \vec{v}$ are two vectors that are not parallel to each other, but parallel to a plane



then any other vector in the plane

$\vec{w}$ can be written as $\vec{w} = t\vec{u} + s\vec{v}$ for two scalars t and s . So if the plane passes through (x_0, y_0, z_0) , the parametric form is given by:

$$\vec{r}(s, t) = \langle x_0, y_0, z_0 \rangle + s\vec{u} + t\vec{v}$$

There are two parameters s and t since the plane is 2-dimensional.
