MAT 203 L22 Tue Nov 25

<u>Recall:</u> • If $\vec{F} = \nabla f$ (that is $\vec{F}$ is conservative w/ potential f), and if C is a curve parametrized by $\vec{r}(t)$, $a \le t \le b$

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

If C is also a closed curve (that it's a "loop"),

$$\int_C \vec{F} \cdot d\vec{r} = 0.$$

• Green's theorem:

Let $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ be a vector field in the plane.

Let C be a simple, closed

(no self-intersections)

curve oriented counterclockwise.
Suppose C bounds a simply connected region D. no holes

Then:

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx\,dy}$$

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§6.5 Divergence & curl

If f is a scalar function, its gradient is $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$, and ∇f is a vector field.
If we instead have a vector field $\vec{F}$, we can construct a scalar function that measures how

much a flow is "flowing outwards".

<u>Def</u>: If $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$ is a vector field <u>the divergence</u> of $\vec{F}$ is

$$\boxed{\nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}}$$

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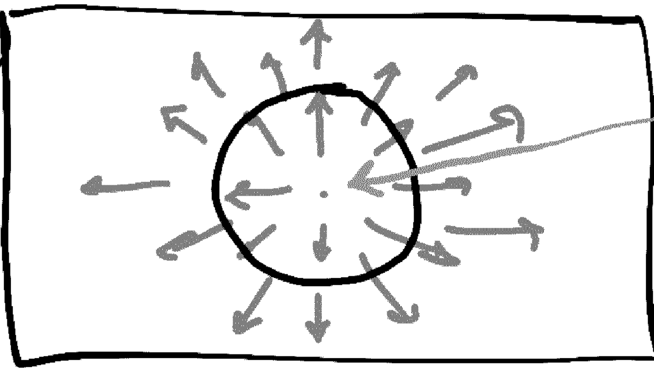
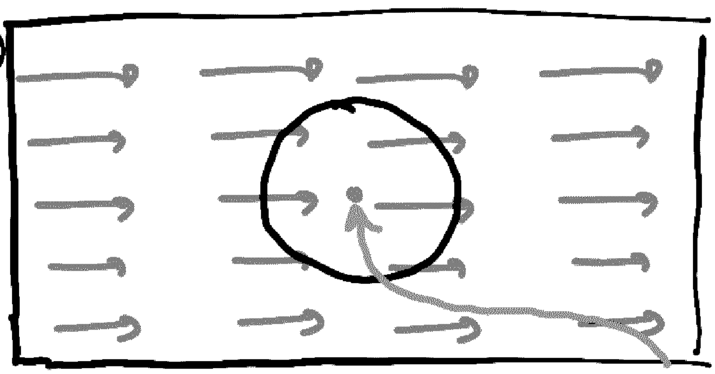
(1)

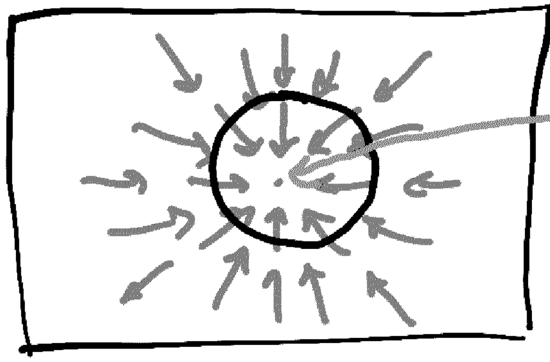


Divergence is >0 here more of the vector field flows outwards at this point.

(2)



Divergence at this point is zero since the vector field flows in & out an "equal amount."

(3)  Divergence at this point is <0 since more of the vector field is flowing in to the center.

<u>Ex</u>: The actual vector fields in 1, 2, 3 above are:

(1) $\vec{F}(x,y) = \langle x, y \rangle$, and we have $(\nabla \cdot \vec{F})(x,y) = 1+1 = 2$.

(2) $\vec{G}(x,y) = \langle 1, 0 \rangle$, and we have $(\nabla \cdot \vec{G})(x,y) = 0+0 = 0$

(3) $\vec{H}(x,y) = \langle -x, -y \rangle$, and we have $(\nabla \cdot \vec{H})(x,y) = -1-1 = -2$.

<u>Ex</u>: Let $\vec{F} = \langle e^x, yz, -yz^2 \rangle$, and let's compute the divergence at $(0,2,-1)$:

$$(\nabla\cdot\vec{F})(0,2,-1) = \frac{\partial e^x}{\partial x}(0,2,-1) + \frac{\partial (yz)}{\partial y}(0,2,-1)$$

$$+ \frac{\partial(-yz^2)}{\partial z}(0,2,-1) = e^0 + (-1) + \underbrace{(-2(2)(-1))}_{=4}$$

$$= 4.$$

<u>Def</u>: Let $\vec{F}(x,y,z) = \langle P,Q,R\rangle$ be a vector field in space. The <u>curl of $\vec{F}$</u> is the vector field

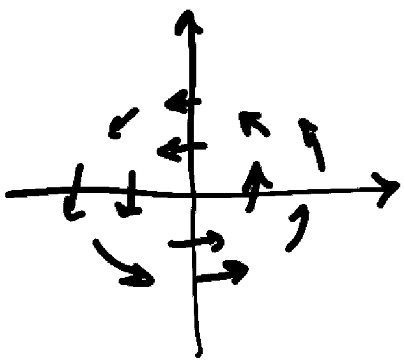
$$(\nabla\times\vec{F})(x,y,z) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$

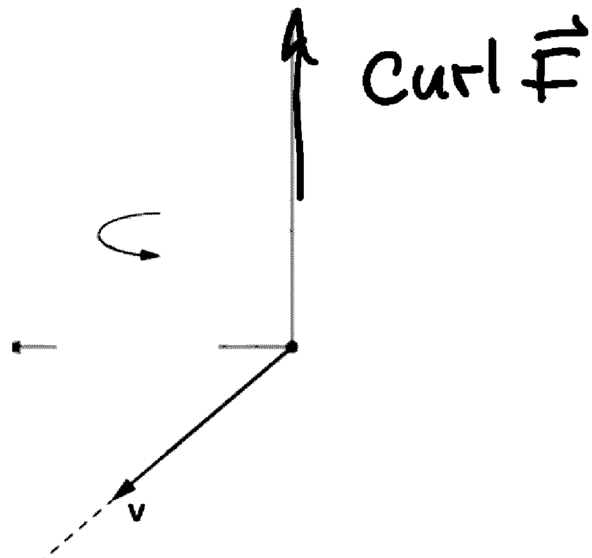
$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\imath} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\hat{\jmath}$$

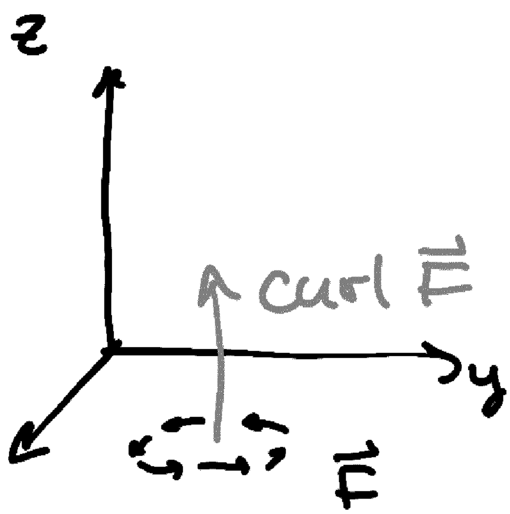
$$+ \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

The curl roughly measures how much the vector field "rotates".

If we consider the vector field $\vec{F}(x,y,z) = \langle -y, x, 0 \rangle$ then in each slice $z = c$ in space, $\vec{F}$ looks like:







<u>Ex</u>: Let $\vec{F}(x,y,z) = \langle x^2 z, e^y + xz, xyz \rangle$.
Then we compute the curl:

$$(\nabla \times F)(x,y,z) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 z & e^y + xz & xyz \end{vmatrix}$$

$$= \langle xz - x, \; x^2 - yz, \; z \rangle$$

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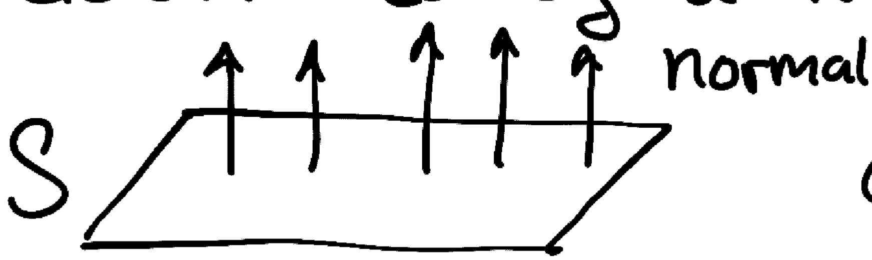
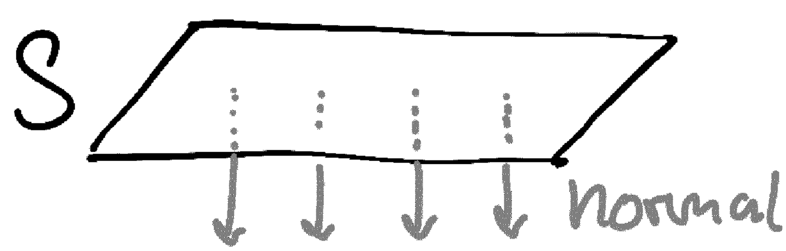
§6.6 Surface integrals

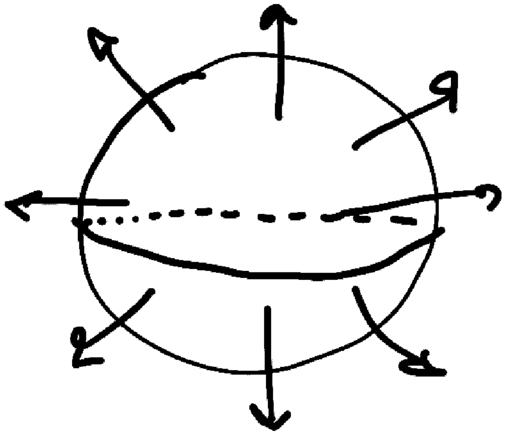
Let S be a surface in space. Much like a curve has a direction, or an orientation, a surface also has an orientation. The orientation of a surface is determined by a normal vector



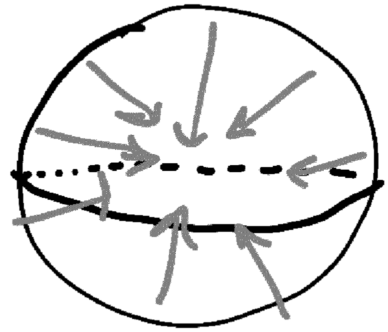
Orientation 1



Orientation 2



"outwards normal"



"inwards normal"

Same surface but different orientations

<u>Def</u>: Let S be a surface in space with parametrization $\vec{r}(s,t)$, $(s,t) \in D$ for some domain D. Let $\vec{F}$ be a vector field.

The surface integral of $\vec{F}$ over S is:

$$\boxed{\iint_S \vec{F} \cdot d\vec{r} = \iint_D \vec{F}(\vec{r}(s,t)) \cdot \vec{N}(s,t)\, ds\, dt}$$

where $\vec{N}(s,t) = \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}$

For a parametrized surface $\vec{r}(s,t)$, the vector $\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}$ is normal to the surface.

Ex: Let $\vec{F}(x,y,z) = \langle -y, x, 0 \rangle$, and let S be the surface parametrized by $\vec{r}(u,v) = \langle u, v^2 - u, u+v \rangle$
$0 \leq u \leq 3$, $0 \leq v \leq 4$. Compute $\iint_S \vec{F} \cdot d\vec{r}$.

Sol: We first compute the normal

$$\frac{\partial \vec{r}}{\partial u} = \langle 1, -1, 1 \rangle$$

$$\frac{\partial \vec{r}}{\partial v} = \langle 0, 2v, 1 \rangle$$

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 0 & 2v & 1 \end{vmatrix} = \langle -1-2v, -1, 2v \rangle$$

Then

$$\iint_S \vec{F}\cdot d\vec{r} = \int_0^4\int_0^3 \vec{F}(\vec{r}(u,v))\cdot\left(\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}\right)du\,dv$$

$$= \int_0^4\int_0^3 \langle -v^2+u,\ u,0\rangle\cdot\langle -1-2v,-1,\ 2v\rangle\,du\,dv$$

$$= \int_0^4\int_0^3 (-v^2+u)(-1-2v)+u(-1)\ du\,dv$$

$$= \int_0^4\int_0^3 v^2+2v^3-u-2uv-u\ du\,dv$$

$$= \int_0^4 \left[(v^2+2v^3)u-(1+v)u^2\right]_0^3 dv$$

$$= \int_0^4 3(v^2+2v^3)-9(1+v)\,dv$$

$$= \left[v^3+\frac{6v^4}{4}-9v-\frac{9v^2}{2}\right]_0^4$$

$$= 4^3+\frac{6\cdot 4^4}{4}-9\cdot 4-\frac{9\cdot 4^2}{2}$$
