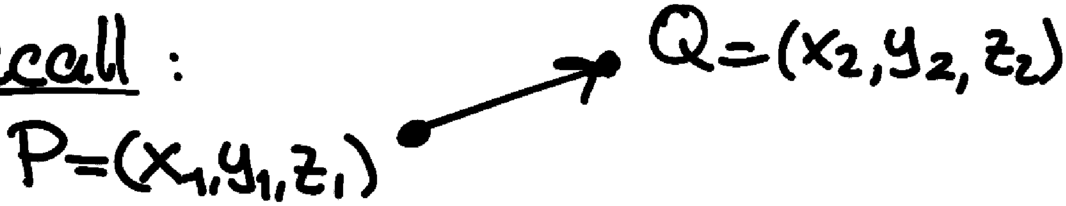
MAT 203 L2 Thu Aug 28

Recall:



$$\vec{v}=\overrightarrow{PQ}=\langle x_2-x_1, y_2-y_1, z_2-z_1\rangle$$
$$=(x_2-x_1)\vec{i}+(y_2-y_1)\vec{j}+(z_2-z_1)\vec{k}$$

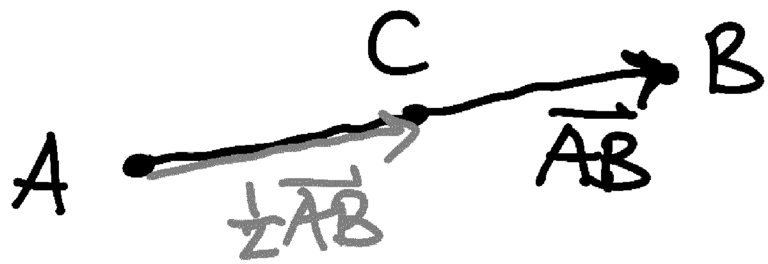
$\vec{i}=\langle 1,0,0\rangle$, $\vec{j}=\langle 0,1,0\rangle$, $\vec{k}=\langle 0,0,1\rangle$

"Standard unit vectors".

$$\|\vec{v}\|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

EX: Let $A=(x_1,y_1,z_1)$, $B=(x_2,y_2,z_2)$. Suppose C is the midpoint of A,B. Express the coordinates of C in terms of those of A and B.

Sol:



$C=(x,y,z)$. We have

$\vec{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$

$\vec{AC} = \langle x - x_1, y - y_1, z - z_1 \rangle$.

__Note__ $\vec{AC} = \frac{1}{2}\vec{AB} = \left\langle \frac{x_2 - x_1}{2}, \frac{y_2 - y_1}{2}, \frac{z_2 - z_1}{2} \right\rangle$

$$\begin{cases} x - x_1 = \dfrac{x_2 - x_1}{2} \\ y - y_1 = \dfrac{y_2 - y_1}{2} \\ z - z_1 = \dfrac{z_2 - y_1}{2} \end{cases}$$

Solve for (x, y, z):

$$\begin{cases} x = \dfrac{x_2 - x_1}{2} + x_1 = \dfrac{x_2 + x_1}{2} \\ y = \dfrac{y_2 + y_1}{2} \\ z = \dfrac{z_2 + z_1}{2} \end{cases}$$

Ans: $C = \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}, \frac{z_2 + z_1}{2} \right)$

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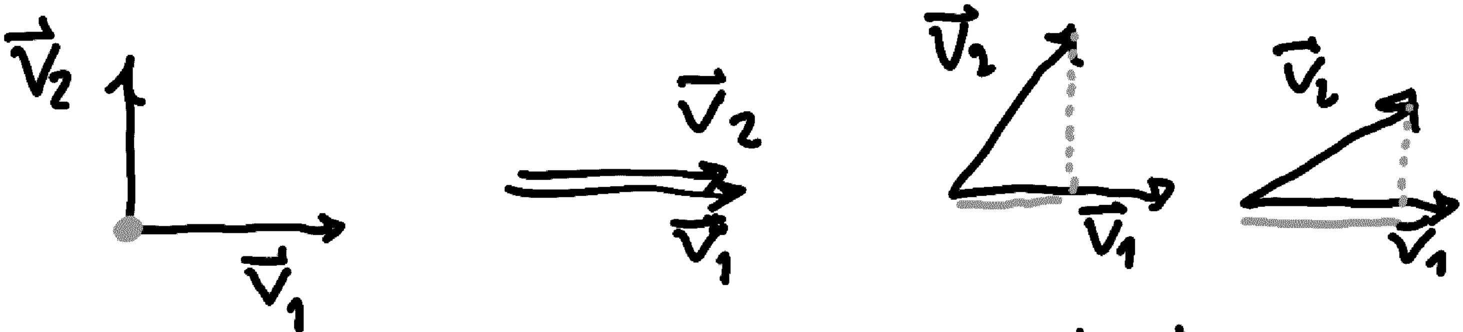
§2.3 Dot product

The dot product of two vectors $\vec{v}_1, \vec{v}_2$ is a real number $\vec{v}_1 \cdot \vec{v}_2$.

Roughly tells us "how much" in the same direction two vectors point.

$\vec{v}_1, \vec{v}_2$ unit vectors $(\|\vec{v}_1\| = \|\vec{v}_2\| = 1)$.



$\vec{v}_1 \cdot \vec{v}_2 = 0$ $\quad$ $\vec{v}_1 \cdot \vec{v}_2 = 1$ $\quad$ $\vec{v}_1 \cdot \vec{v}_2 = \frac{1}{2}$ $\quad$ $\vec{v}_1 \cdot \vec{v}_2 = \frac{3}{4}$.

<u>Definition</u>: Let $\vec{v}_1 = \langle x_1, y_1, z_1 \rangle$,
$\vec{v}_2 = \langle x_2, y_2, z_2 \rangle$.

Then $\vec{v}_1 \cdot \vec{v}_2 = x_1x_2 + y_1y_2 + z_1z_2$.

<u>Ex</u>: $\langle 1,0 \rangle \cdot \langle 0,1 \rangle = 1\cdot 0 + 0 \cdot 1 = 0$

$\langle 1,0 \rangle \cdot \langle 1,0 \rangle = 1\cdot 1 + 0\, 0 = 1$

$\langle 1,0 \rangle \cdot \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle = 1 \cdot \frac{1}{2} + 0 \cdot \frac{\sqrt{3}}{2} = \frac{1}{2}$

$\langle 1,0 \rangle \cdot \langle \frac{3}{4}, \frac{\sqrt{7}}{4} \rangle = 1 \cdot \frac{3}{4} + 0 \cdot \frac{\sqrt{7}}{4} = \frac{3}{4}$.

Properties:

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors and let c be a scalar.

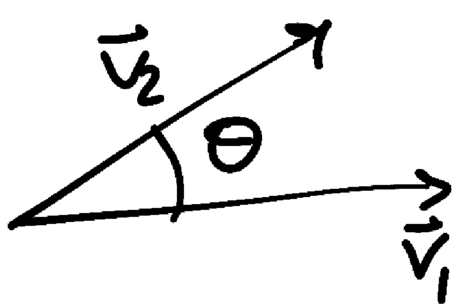
(i) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

(ii) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

(iii) $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$

(iv) $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

The dot product is, as illustrated above, related to the angle between two vectors.



If $\vec{v}_1, \vec{v}_2$ are any two vectors, and $0 \le \theta \le \pi$ is the smallest angle between them, then

$$\boxed{\vec{v}_1 \cdot \vec{v}_2 = \|\vec{v}_1\| \cdot \|\vec{v}_2\| \cdot \cos\theta.}$$

This allows us to compute the angle:

$$\theta = \arccos\left(\frac{\vec{V}_1 \cdot \vec{V}_2}{\|\vec{V}_1\| \, \|\vec{V}_2\|}\right).$$

<u>Ex</u>: $\vec{V}_1 = \langle 1,1,1 \rangle$, $\vec{V}_2 = \langle 2,-1,-3 \rangle$ then we find the angle b/w them:

$$\vec{V}_1 \cdot \vec{V}_2 = 1\cdot 2 + 1\cdot(-1) + 1\cdot(-3) = 2-1-3 = -2$$

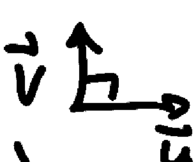
$$\|\vec{V}_1\| = \sqrt{1^2+1^2+1^2} = \sqrt{3}, \ \|\vec{V}_2\| = \sqrt{2^2+(-1)^2+(-3)^2}$$

$$= \sqrt{4+1+9} = \sqrt{14}$$

$$\cos\theta = \frac{\vec{V}_1 \cdot \vec{V}_2}{\|\vec{V}_1\| \, \|\vec{V}_2\|} = \frac{-2}{\sqrt{3}\cdot\sqrt{14}} = -\frac{2}{\sqrt{42}}$$

$$\theta = \arccos\left(-\frac{2}{\sqrt{42}}\right) (\approx 108^\circ).$$

<u>Def</u>: We say that $\vec{u}$ and $\vec{v}$ are <u>orthogonal</u> if $\vec{u} \cdot \vec{v} = 0$.



(Corresponds to angle $\theta = \pi$.)

Ex, For what value of x is $\vec{u}=\langle 2,8,-1\rangle$ orthogonal to $\vec{v}=\langle x,-1,2\rangle$?

Sol: They are orthogonal if $\vec{u}\cdot\vec{v}=0$

$$\vec{u}\cdot\vec{v}=\langle 2,8,-1\rangle\cdot\langle x,-1,2\rangle=2x+8(-1)+(-1)\cdot 2$$
$$=2x-8-2=2x-10=0$$
$$\iff 2x=10 \iff \boxed{x=5}$$

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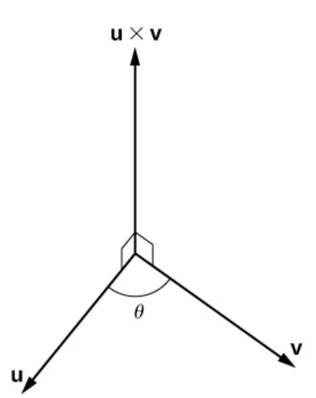
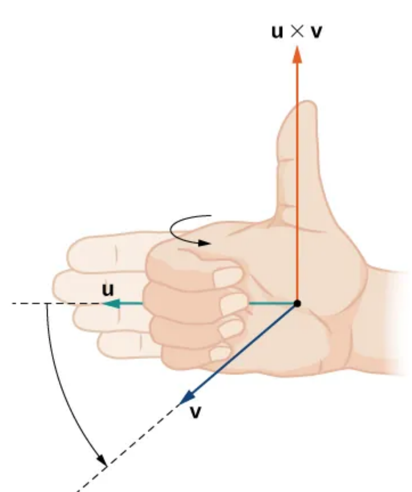
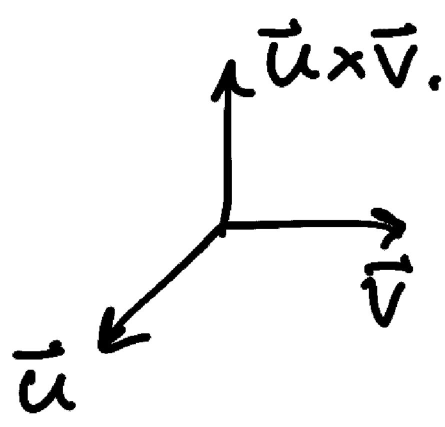
§2.4 Cross product

This only works in space and not in the plane.

The cross product of two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$ is a vector $\vec{u}\times\vec{v}$ that is orthogonal to both $\vec{u}$ and $\vec{v}$.







The right hand rule determines which direction $\vec{u} \times \vec{v}$ should point.

<u>Definition</u>: Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ $\vec{v} = \langle v_1, v_2, v_3 \rangle$. Then

$$\vec{u} \times \vec{v} = (u_2 v_3 - u_3 v_2)\vec{i} - (u_1 v_3 - u_3 v_1)\vec{j} + (u_1 v_2 - u_2 v_1)\vec{k}.$$

Let's check that $\vec{u}$ is orthogonal to $\vec{u} \times \vec{v}$, namely that $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$.

$$\vec{u} \cdot (\vec{u} \times \vec{v}) = u_1(u_2 v_3 - u_3 v_2) - u_2(u_1 v_3 - u_3 v_1) + u_3(u_1 v_2 - u_2 v_1) = \underline{u_1 u_2 v_3} - \underline{u_1 u_3 v_2} - \underline{u_1 u_2 v_3} + \underline{u_2 u_3 v_1} + \underline{u_1 u_3 v_2} - \underline{u_2 u_3 v_1} = 0$$

Can similarly check $\vec{v} \cdot (\vec{u} \times \vec{v}) = 0$.

<u>Ex</u>: Let $\vec{p} = \langle 5, 1, 2 \rangle$, $\vec{q} = \langle -2, 0, 1 \rangle$ then

$$\vec{p} \times \vec{q} = (1 \cdot 1 - 2 \cdot 0)\vec{i} - (5 \cdot 1 - 2 \cdot (-2))\vec{j} + (5 \cdot 0 - 1 \cdot (-2))\vec{k} = \vec{i} - 9\vec{j} + 2\vec{k} = \langle 1, -9, 2 \rangle$$

Properties: $\vec{u}, \vec{v}, \vec{w}$ vectors, c scalar

(i) $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

(ii) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

(iii) $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$

(iv) $\vec{u} \times \vec{0} = \vec{0} \times \vec{u} = \vec{0}$

(v) $\vec{v} \times \vec{v} = \vec{0}$

(vi) $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$

Quick guide to determinants:

The formula for the cross product is hard to memorize but is easier when using "determinants."

2x2 determinants:

a, b, c, d scalars, then

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb \text{ by def.}$$

3x3. $a, b, \ldots, i$ scalars, then

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \qquad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix}$$

$$+ c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$= a(ei - hf) - b(di - gf) + c(dh - ge)$$

We can then reformulate the formula for the cross product as follows:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \vec{i} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \vec{j} \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}$$

$$+ \vec{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = (u_2 v_3 - v_2 u_3)\vec{i}$$

$$-(u_1 v_3 - v_1 u_3)\vec{j} + (u_1 v_2 - v_1 u_2)\vec{k}.$$

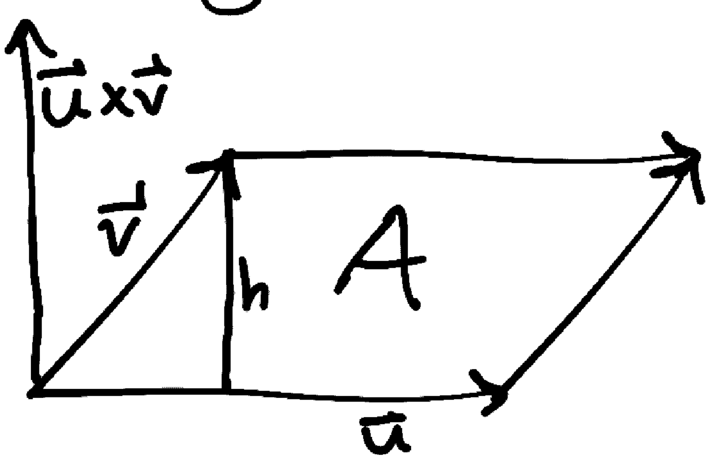
<u>Ex</u>: $\vec{p} = \langle -1, 2, 5\rangle$, $\vec{q} = \langle 4, 0, -3\rangle$

$$\vec{p}\times\vec{q} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 5 \\ 4 & 0 & -3 \end{vmatrix} = \vec{i}\begin{vmatrix} 2 & 5 \\ 0 & -3\end{vmatrix} - \vec{j}\begin{vmatrix} -1 & 5 \\ 4 & -3\end{vmatrix}$$

$$+\vec{k}\begin{vmatrix} -1 & 2 \\ 4 & 0\end{vmatrix} = (-6-0)\vec{i} - (3-20)\vec{j} + (0-8)\vec{k}$$

$$= -6\vec{i} + 17\vec{j} - 8\vec{k}$$

Geometrically, it turns out that $\|\vec{u}\times\vec{v}\|$ is the area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$.



$\|\vec{u}\times\vec{v}\| = A.$

<u>Also</u>: $\|\vec{u}\times\vec{v}\| = \underbrace{\|\vec{u}\|}_{\text{base}} \cdot \underbrace{\|\vec{v}\|\cdot\sin\theta}_{\text{height of parallelogram}}.$
