

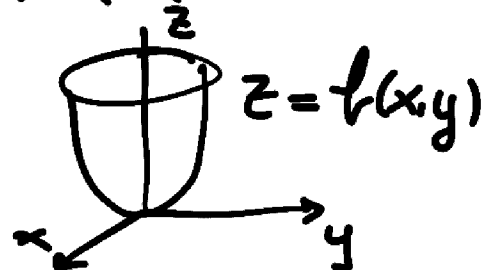
§ 6.1 Vector fields

Definition: A vector field is an assignment of a vector at every point in the plane, or in space.

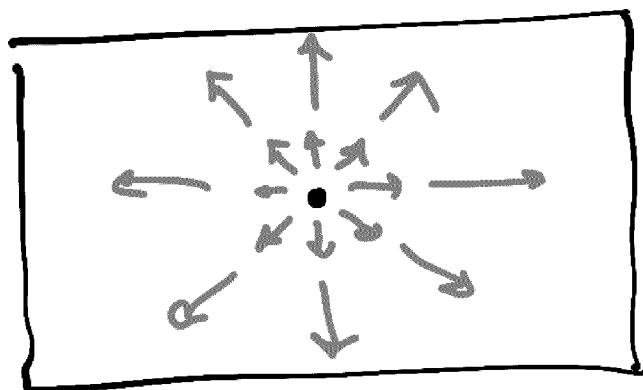
Ex: Gradient vector field of a function $f(x,y)$.

eg. $f(x,y) = x^2 + y^2$

$$\nabla f(x,y) = \langle 2x, 2y \rangle.$$

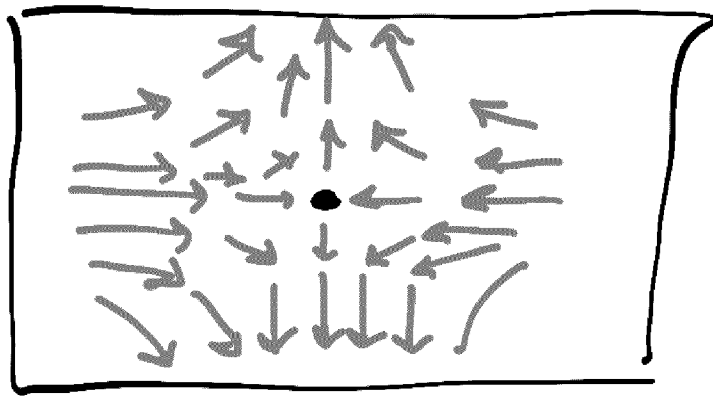
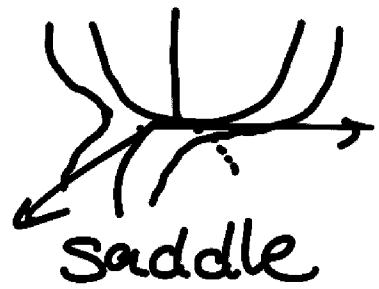


So in the plane we draw the vector $\langle 2x, 2y \rangle$ at position (x,y) :
xy-plane



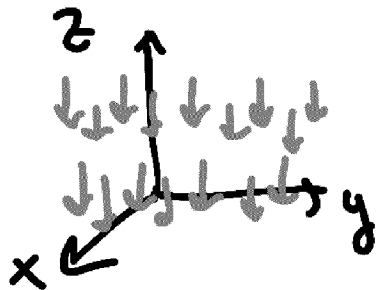
eg. $f(x,y) = x^2 - y^2$

$\nabla f(x,y) = \langle 2x, -2y \rangle$



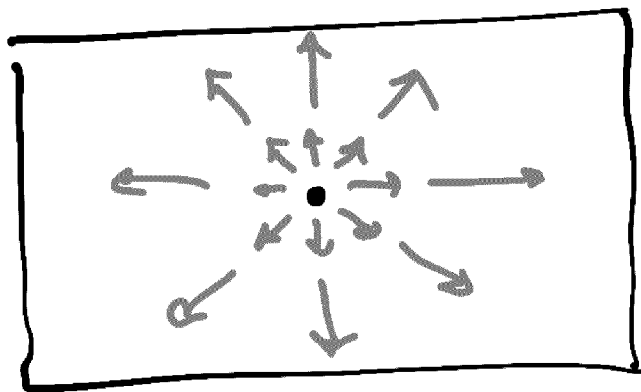
xy-plane

Ex: Gravitational vector field:
It's a vector field in space that
assigns every pt to the vector
 $\langle 0, 0, -g \rangle$ ($g \approx 9.82 \text{ m/s}^2$)

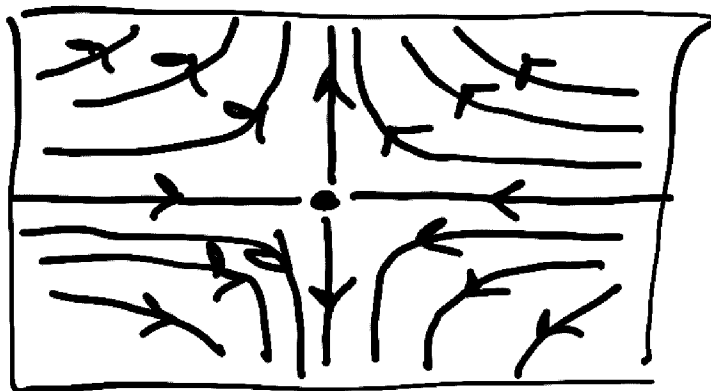
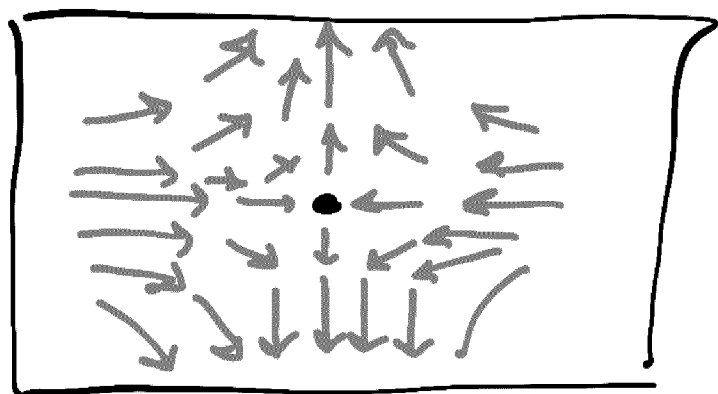
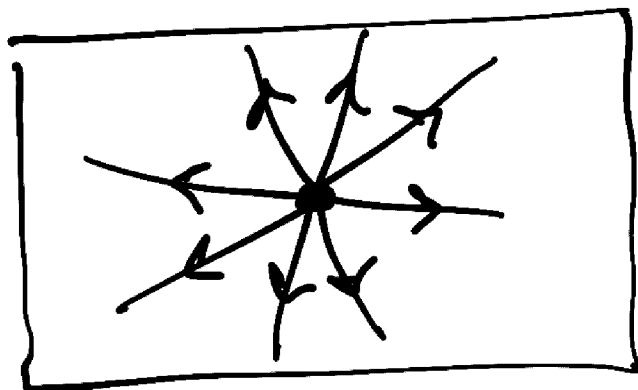


Definition: A flow line of
a vector field is a curve that
is tangent to the vector field.

Ex. Vector field



flow lines



Conservative vector fields :

A vector field $\vec{F}$ is called "conservative" if

$\vec{F} = \nabla f$ for some function f .

If $\vec{F}$ is conservative, such a function is called a potential function.

We will later see why this is important.

Ex: $f(x, y, z) = x^2yz - \sin(xy)$ is a potential function for

$$\vec{F}(x, y, z) = \langle 2xyz - y\cos(xy), x^2z - x\cos(xy), x^2y \rangle$$

Since we may compute ∇f , and we get

$$\begin{aligned}\nabla f &= \langle 2xyz - y\cos(xy), x^2z - x\cos(xy), x^2y \rangle \\ &= \vec{F}.\end{aligned}$$

Theorem: If f and g are potential functions for a vector field $\vec{F}$, then $f = g + \text{Constant}$.

Ex, Find a potential function for $\vec{F}(x, y, z) = \langle 0, 0, -g \rangle$

By inspection we see that

$f(x,y,z) = -gz$ is a potential.

So every potential is $-gz + C$, where C is a constant.

Ex. Find a potential for

$$\vec{F}(x,y) = \langle y, x + \cos(y) \rangle.$$

Sol. If there's a potential

$f(x,y)$, then

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle y, x + \cos(y) \rangle$$

$$\begin{cases} \frac{\partial f}{\partial x} = y \\ \frac{\partial f}{\partial y} = x + \cos(y) \end{cases}$$

we now
solve this
system.

1st eq gives $f(x,y) = xy + g(y)$

then $\frac{\partial f}{\partial y} = x + g'(y) = x + \cos(y)$

$$\Leftrightarrow g'(y) = \cos(y) \Leftrightarrow g(y) = \sin(y) + C$$

So $f(x,y) = xy + \sin(y) + C$
is a potential.

Ex, Find a potential for

$$\vec{F}(x,y,z) = \langle 7, -2, x^3 \rangle.$$

Sol: Like above we try to
solve $\nabla f = \vec{F}$.

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle 7, -2, x^3 \rangle$$

$$\begin{cases} \partial f / \partial x = 7 \\ \partial f / \partial y = -2 \\ \partial f / \partial z = x^3 \end{cases}$$

$$1^{\text{st}} \text{ eq} \Rightarrow f(x,y,z) = 7x + g(y,z).$$

Inputting it into 2nd eq gives

$$\frac{\partial g}{\partial y} = -2 \Rightarrow g(y,z) = -2y + h(z)$$

Now inserting it into 3rd equ

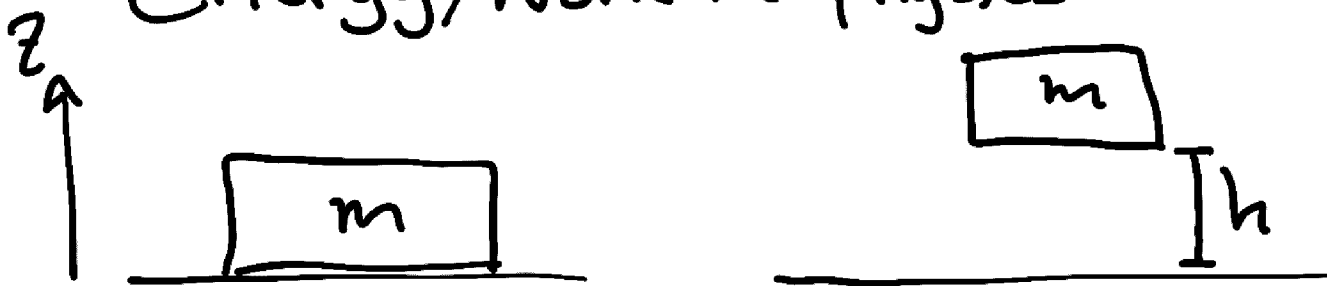
$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z} (7x - 2y + h(z)) = \boxed{h'(z) = x^3}$$

Since left hand side only depends on z , and right hand side only depends on x , they can not be equal for all (x, y, z) .

Conclusion: There is no potential!

§6.2 Line integrals

Energy/Work in physics.



box w/ mass m

The energy required to lift the mass a height is

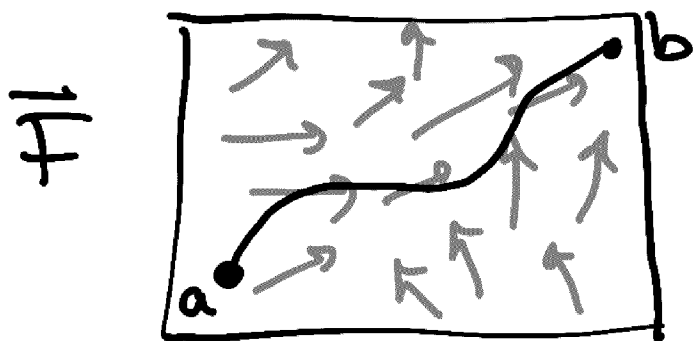
$$E = mgh.$$

Another way to phrase this is as an integral:

$$\int_0^h mg \, dz = \int_0^h \text{Force} \, dz$$

This kind of integral is the special case of a more general integral:

If $\vec{F}$ is a vector field in the plane (let's say),



then if C is a curve in the plane

one can compute the energy it requires to walk from a to b .

How do we do it?

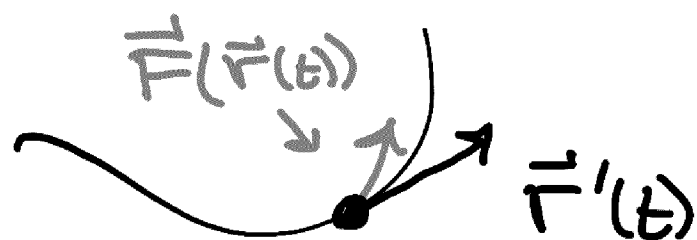
Let's parametrize C :

$$\vec{r}(t), \quad t_0 \leq t \leq t_1$$

So that
$$\begin{cases} \vec{r}(t_0) = a \\ \vec{r}(t_1) = b \end{cases}$$

Then Energy = $\int_{t_0}^{\quad} \text{Force}(\vec{r}(t)) dt$

but what is Force at $\vec{r}(t)$?



Force($\vec{r}(t)$) = $\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)$. So

$$\text{Energy} = \int_{t_0}^{\quad} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt.$$

Now, we can write this as

$$\text{Energy} = \int_C \vec{F} \cdot d\vec{r}$$

This is an example of a line integral.