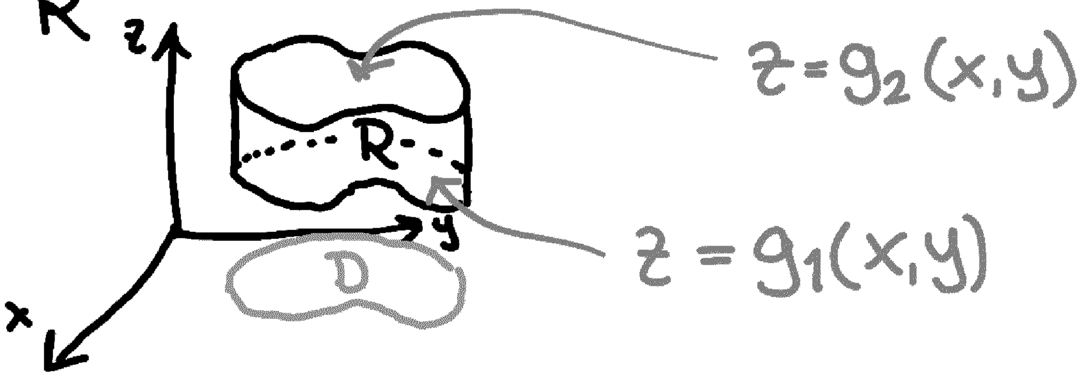
MAT 203 L18 Tue Nov 4

<u>Recall</u>: Triple integrals

If $R=[a,b]\times[c,d]\times[e,f]$ is a rectangular box, then

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \int_e^f\int_c^d\int_a^b f(x,y,z)\,dx\,dy\,dz$$



Let R be $\{(x,y,z)\mid (x,y)\in D,$
$g_1(x,y)\leq z\leq g_2(x,z)\}$

$$\iiint_R f(x,y,z)\,dx\,dy\,dz$$

$$= \iint_D\left(\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\right)dx\,dy.$$

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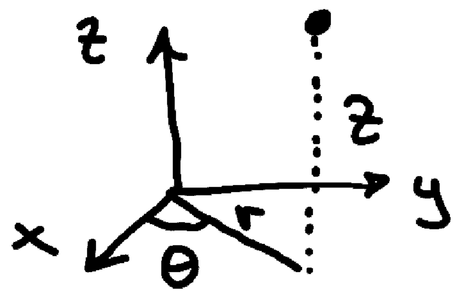
§5.5 <u>Cylindrical and spherical coords.</u>

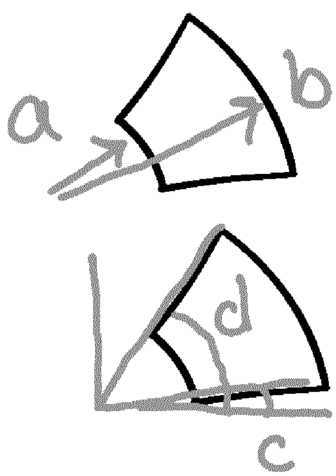
Cylindrical coords: (r,θ,z)

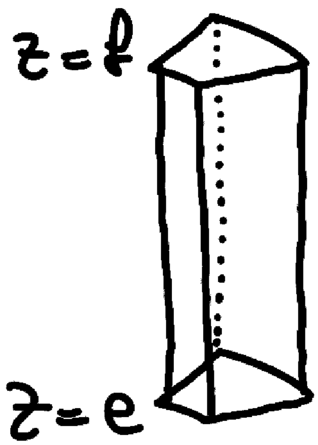
$$\begin{cases} x = r\cos\theta & 0 \le r < \infty \\ y = r\sin\theta & 0 \le \theta \le 2\pi \\ z = z & -\infty < z < \infty \end{cases}$$



To change variables to cylindrical coords, we should find dV in terms of dr, $d\theta$, and dz.

Rectangle in cylindrical coords:





$a \le r \le b$

$c \le \theta \le d$

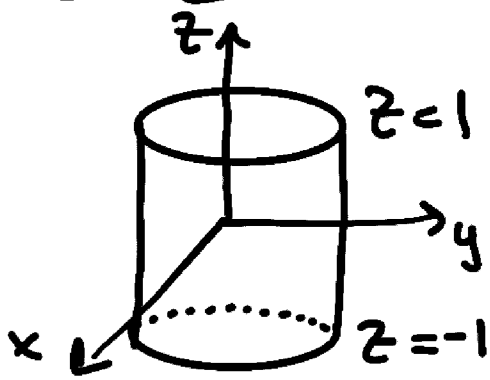
$e \le z \le f$

Recall that the area element of the annular sector is $dA = r\,dr\,d\theta$. So the volume is

$$dV = dA\,dz = r\,dr\,d\theta\,dz$$

Ex: Compute $\iiint_C yz\,dx\,dy\,dz$, where C is the cylinder w/ base the unit disk, and $-1 \le z \le 1$.

Sol:



In cylindrical coords,

$$C = \{(r,\theta,z) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ -1 \le z \le 1\}$$

$$\iiint_C yz\,dx\,dy\,dz = \int_{-1}^{1}\int_{0}^{2\pi}\int_{0}^{1} z\,r\sin\theta\, r\,dr\,d\theta\,dz$$

$$= \left(\int_{-1}^{1} z\,dz\right)\left(\int_{0}^{1} r^2\,dr\right)\left(\int_{0}^{2\pi}\sin\theta\,d\theta\right)$$

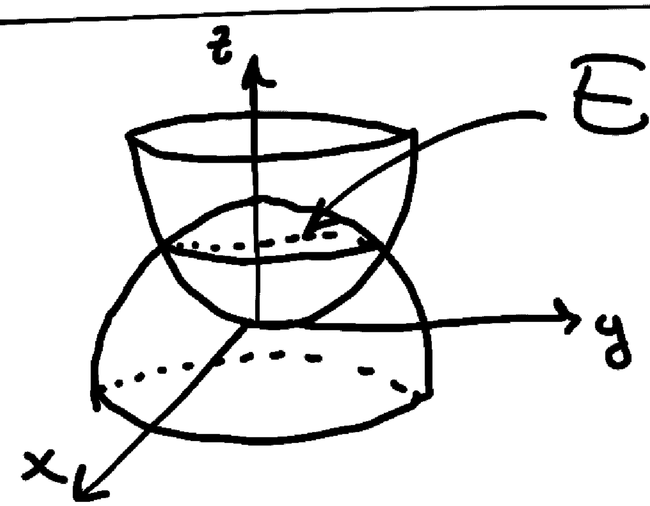
$$= \left[\frac{z^2}{2}\right]_{-1}^{1}\left[\frac{r^3}{3}\right]_{0}^{1}\left[-\cos\theta\right]_{0}^{2\pi}$$

$= 0 \cdot 1 \cdot (-1-(-1)) = 0$

<u>Ex</u>: Compute $\iiint_E xy \, dx dy dz$

Where E is the region between the graphs $z = x^2+y^2$ and $z = 8 - x^2 - y^2$

<u>Sol</u>.



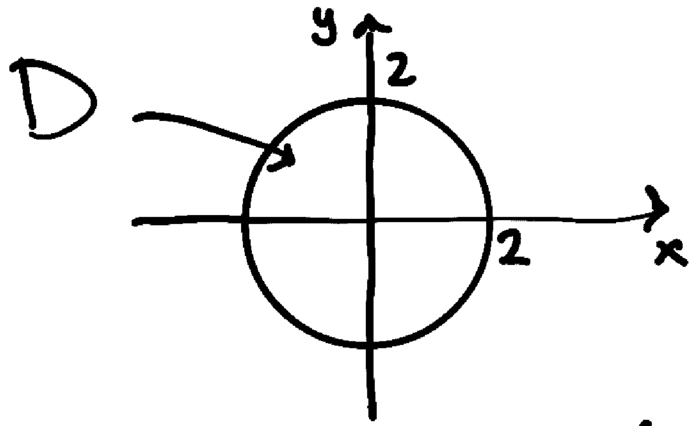
Intersection between the graphs:

$$x^2+y^2 = z = 8 - x^2 - y^2$$

$$\Leftrightarrow x^2+y^2 = 4$$

Shadow of E in the xy-plane:

$$D = \{(x,y) \mid x^2+y^2 \leq 4\}$$



$$\iiint_E xy\,dx\,dy\,dz = \iint_D \left(\int_{x^2+y^2}^{8-x^2-y^2} xy\,dz \right) dx\,dy$$

$$= \iint_D xy\left(8-2x^2-2y^2\right) dx\,dy$$

Now let's change to polar coords:

$$D = \{(r,\theta) \mid 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$$

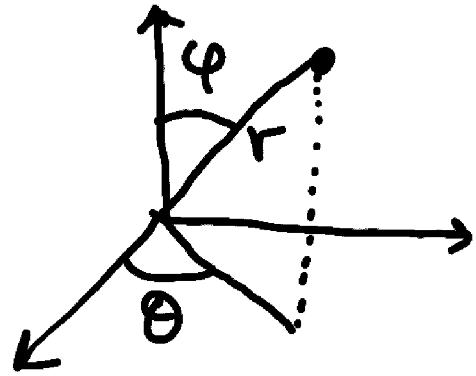
$$\int_0^{2\pi}\int_0^{2} r^2 \sin\theta\cos\theta\left(8-2r^2\right) r\,dr\,d\theta$$

$$= \left(\int_0^2 r^3\left(8-2r^2\right) dr\right)\left(\int_0^{2\pi} \underbrace{\sin\theta\cos\theta}_{=\frac{1}{2}\sin(2\theta)}\,d\theta\right)$$

$$= \left[2r^4 - \frac{r^6}{3}\right]_0^2 \underbrace{\left[\frac{-\cos(2\theta)}{4}\right]_0^{2\pi}}_{=0} = 0$$

Spherical coords: (ρ, θ, φ)

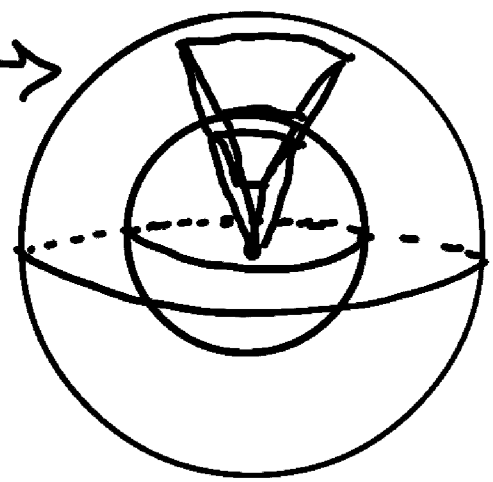
$$\begin{cases} x = r\cos\theta\sin\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\varphi \end{cases}$$



$0 \le r < \infty$

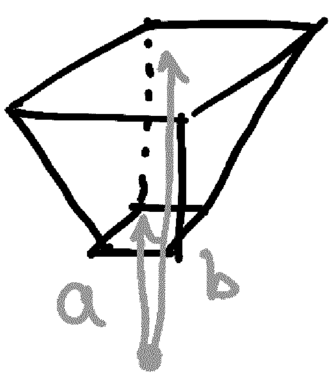
$0 \le \theta \le 2\pi$

$0 \le \varphi \le \pi$

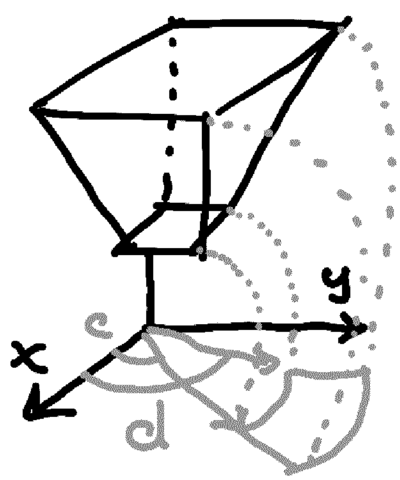


A rectangle in spherical coords is a spherical sector.

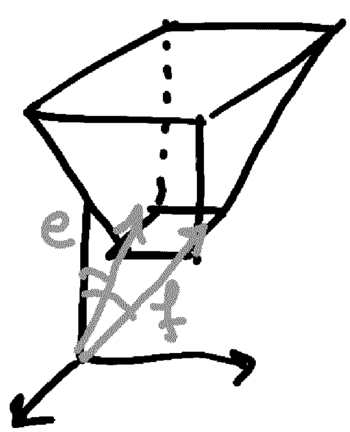
$a \le r \le b$



$c \le \theta \le d$



$e \le \varphi \le f$

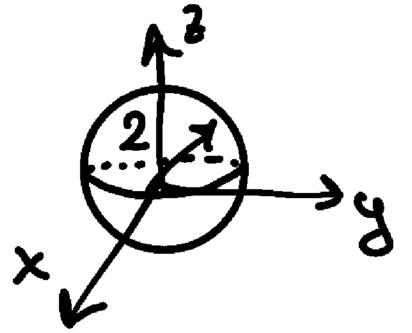


$$\boxed{dV = r^2 \sin\varphi \, dr \, d\theta \, d\varphi}$$

Ex: Compute $\iiint_S z \, dx\,dy\,dz$

Where S is the inside of the sphere, centered at (0,0,0), with radius 2.

Sol:



In spherical coords:

$$S = \{(r,\theta,\varphi) \mid 0 \le r \le 2,\ 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi\}.$$

$$\iiint_S z \, dx\,dy\,dz = \int_0^{\pi}\int_0^{2\pi}\int_0^{2} r\cos\varphi \,(r^2 \sin\varphi)\, dr\, d\theta\, d\varphi$$

$$= \left(\int_0^2 r^3 dr\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin\varphi\cos\varphi \, d\varphi\right)$$

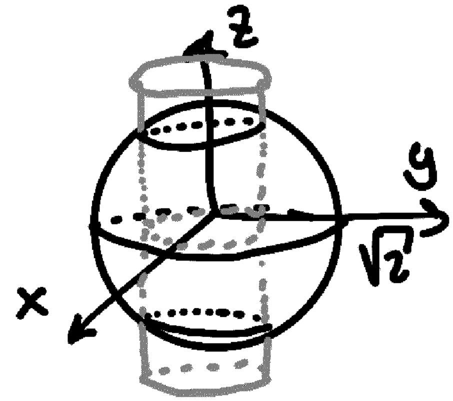
$$= \left[\frac{r^4}{4}\right]_0^2 (2\pi) \int_0^{\pi} \frac{1}{2}\sin(2\varphi)\, d\varphi$$

$$= 8\pi \left[-\frac{\cos(2\varphi)}{4}\right]_0^{\pi} = 8\pi\left(-\frac{1}{4} - \left(-\frac{1}{4}\right)\right) = 0$$

Ex: Let E be the region between the cylinder $x^2+y^2=1$ and outside the sphere $x^2+y^2+z^2=2$. Compute

$$\iiint_S \frac{1}{x^2+y^2+z^2}\,dx\,dy\,dz.$$

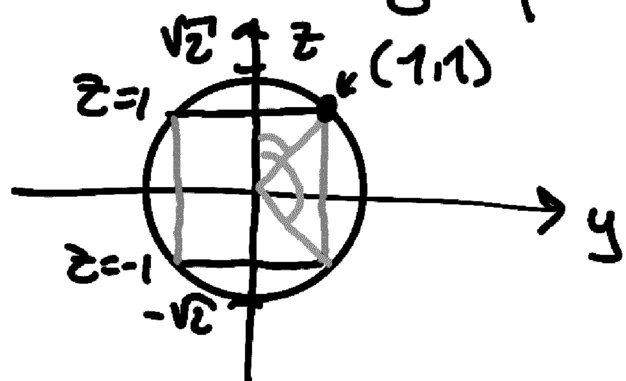
Sol: We first want to describe E using spherical coords.



The bounds on θ is $0 \le \theta \le 2\pi$. To find bounds on φ, let's find the intersection circles of the cylinder and the sphere.

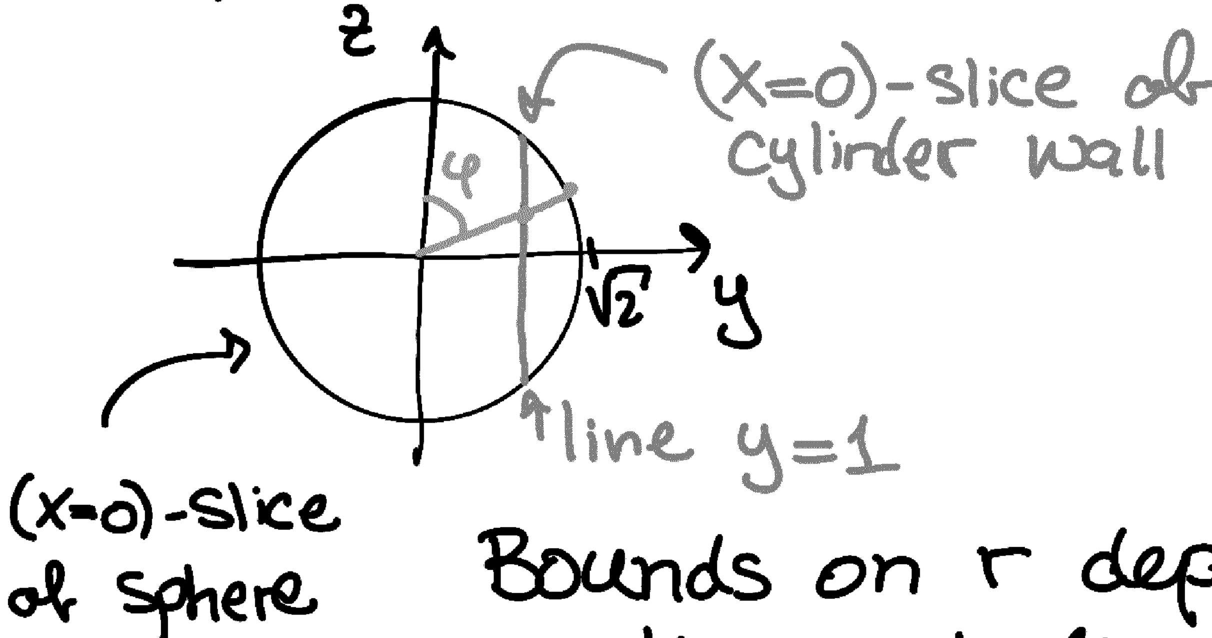
$$\begin{cases} x^2+y^2+z^2=2 \\ x^2+y^2=1 \end{cases} \Rightarrow z^2=1 \Rightarrow z=\pm 1.$$

so in the yz-plane it looks line:



Smaller angle $= \frac{\pi}{4}$
larger angle $= \frac{3\pi}{4}$ so $\frac{\pi}{4} \le \varphi \le \frac{3\pi}{4}$.

Now, we need to find r.



Bounds on r depend on the angle φ.

For $\frac{\pi}{4} \le \varphi \le \frac{3\pi}{4}$, r goes between the line $y=1$ and $r=\sqrt{2}$.

The line $y=1$ in the plane is described by $y = r\sin\varphi = 1$
$\iff r = \frac{1}{\sin\varphi}$.

So $\frac{1}{\sin\varphi} \le r \le \sqrt{2}$.

The integral is therefore:

$$\iiint_S \frac{1}{x^2+y^2+z^2}\,dx\,dy\,dz$$

$$= \iiint_S \frac{r^2\sin\varphi}{r^2}\,dr\,d\theta\,d\varphi$$

$$= \int_0^{2\pi}\int_{\pi/4}^{3\pi/4}\left(\int_{1/\sin\varphi}^{\sqrt{2}} dr\right)\sin\varphi\,d\varphi\,d\theta$$

$$= \int_0^{2\pi}\int_{\pi/4}^{3\pi/4}\left(\sqrt{2}-\frac{1}{\sin\varphi}\right)\sin\varphi\,d\varphi\,d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_{\pi/4}^{3\pi/4} \sqrt{2}\sin\varphi - 1\,d\varphi\right)$$

$$= 2\pi\left[-\sqrt{2}\cos\varphi - \varphi\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$$

$$= 2\pi\left(\left(\frac{\sqrt{2}}{\sqrt{2}}-\frac{3\pi}{4}\right)-\left(-\frac{\sqrt{2}}{\sqrt{2}}-\frac{\pi}{4}\right)\right)$$

$$= 2\pi\left(2-\frac{\pi}{2}\right).$$