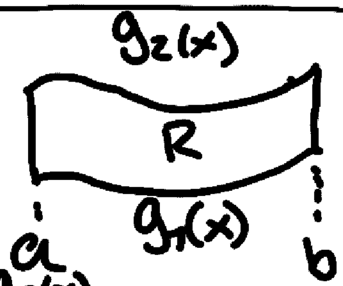
MAT 203 L17 Thu Oct 30

Recall:

- (1)

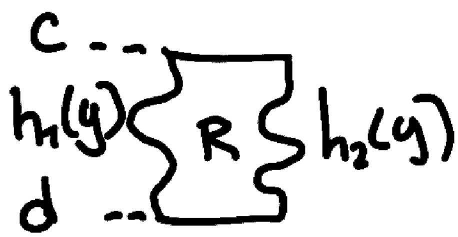


$a \le x \le b$

$g_1(x) \le y \le g_2(x)$

$$\iint_R f(x,y)\,dx\,dy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

(2)



$h_1(y) \le x \le h_2(y)$

$c \le y \le d$

$$\iint_R f(x,y)\,dx\,dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

- Polar Coord change:

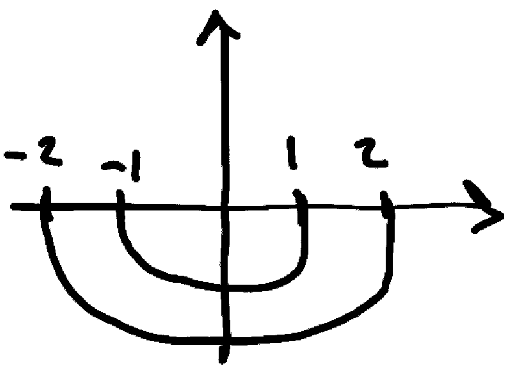
$$dA = dx\,dy = r\,dr\,d\theta.$$

$$\iint_R f(x,y)\,dx\,dy = \iint_R f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Ex: Compute

$$\iint_R x^2+xy \, dx dy \quad \text{where}$$

$$R=\{(r,\theta) \mid 1\leq r\leq 2, \ \pi\leq\theta\leq 2\pi\}$$



$$x^2+xy$$

$$=(r\cos\theta)^2+(r\cos\theta)(r\sin\theta)$$

$$= r^2\cos^2\theta + r^2\cos\theta\sin\theta$$

$$\int_{\pi}^{2\pi}\int_{1}^{2} r^2(\cos^2\theta+\cos\theta\sin\theta)\, r\, dr\, d\theta$$

$$=\left(\int_{\pi}^{2\pi}\cos^2\theta+\cos\theta\sin\theta \, d\theta\right)\left(\int_1^2 r^3 dr\right)$$

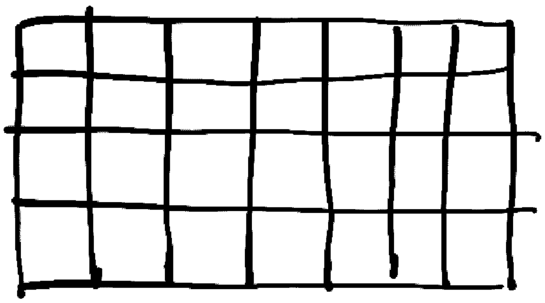
$$=\left[\frac{r^4}{4}\right]_1^2\left(\int_{\pi}^{2\pi}\frac{1+\cos(2\theta)}{2}+\frac{1}{2}\sin(2\theta)\, d\theta\right)$$

$$=\left(4-\frac{1}{4}\right)\cdot\left[\theta+\frac{\sin(2\theta)}{4}-\frac{\cos(2\theta)}{4}\right]_{\pi}^{2\pi}$$

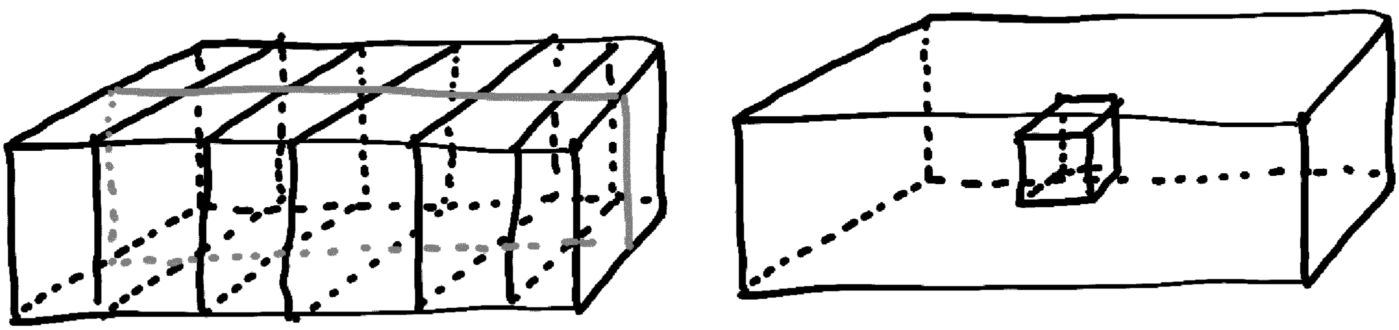
$$=\frac{15}{16}\left(\left(2\pi+0-\frac{1}{4}\right)-\left(\pi+0-\frac{1}{4}\right)\right)=\frac{15\pi}{16}$$

§5.4 Triple integrals:

Similar to how we defined double integrals by subdividing rectangles



we may define triple integrals by subdividing rectangular boxes:



$$\iiint_R f(x,y,z)\, dV, \quad dV = dx\,dy\,dz.$$

The triple integral satisfies the same properties as the double integral. Ex: Compute $\iiint_B x^2yz\, dx\,dy\,dz$

where

$$B = \{(x,y,z) \mid -2 \le x \le 1,\ 0 \le y \le 3,\ 1 \le z \le 5\}$$

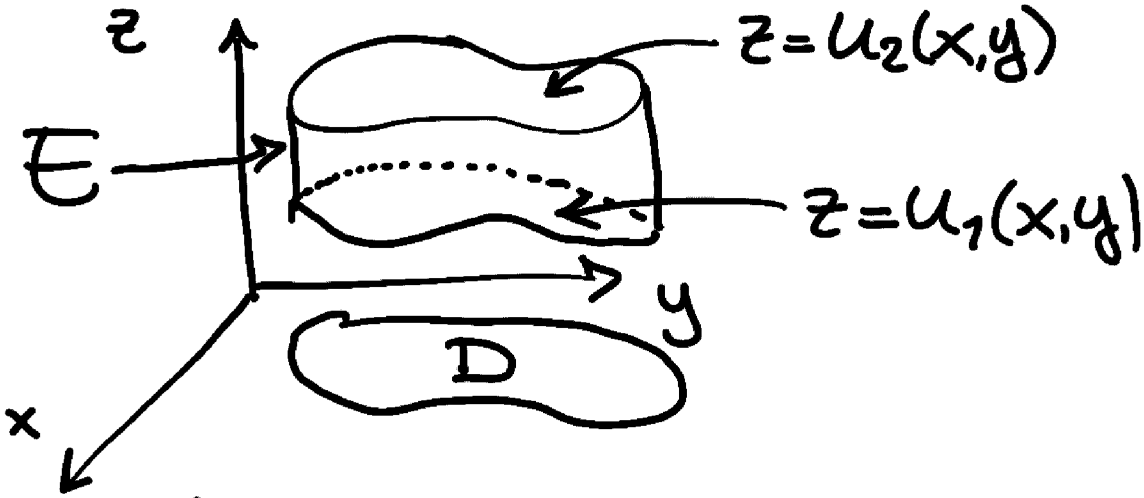
$$\int_{x=-2}^{1} \int_{y=0}^{3} \int_{z=1}^{5} x^2 y z \, dz \, dy \, dx$$

$$= \int_{x=-2}^{1} \int_{y=0}^{3} \left[\frac{x^2 y z^2}{2} \right]_1^5 dy \, dx = \int_{-2}^{1} \int_{0}^{3} 12 x^2 y \, dy \, dx$$

$$= \int_{-2}^{1} \left[6x^2 y^2 \right]_0^3 dx = \int_{-2}^{1} 54 x^2 \, dx = \left[18x^3 \right]_{-2}^{1}$$

$$= 18(1+8) = 162$$

Similar to double integrals, we can also integrate more general regions bounded by graphs:



If $E = \{(x,y,z) \mid (x,y) \in D$
$u_1(x,y) \le z \le u_2(x,y)\}$

then

$$\iiint_E f(x,y,z)\,dx\,dy\,dz = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right) dx\,dy.$$

It works in a similar way when we integrate over a region in space of points (x,y,z) such that

$(x,z) \in D$, $v_1(x,z) \le y \le v_2(x,z)$

or

$(y,z) \in D$, $w_1(y,z) \le x \le w_2(y,z)$.

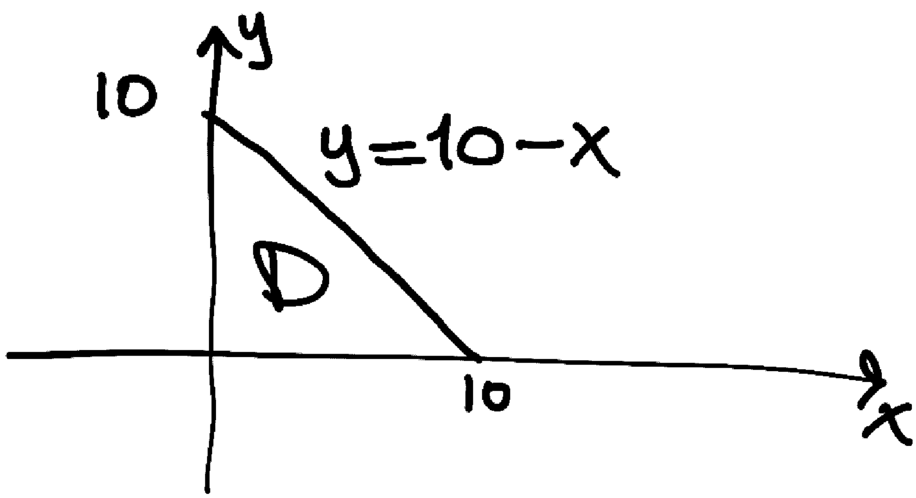
Ex. Compute $\iiint_E x \; dxdydz$, where

$E = \{(x,y,z) \mid y \geq 0,\ x \geq 0,\ x+y \leq 10$
$0 \leq z \leq 10-x-y\}.$

This region consists of those (x,y,z) such that

$(x,y) \in D$, $0 \leq z \leq 10-x-y$, where

$D = \{(x,y) \mid 0 \leq x \leq 10,\ 0 \leq y \leq 10-x\}$



$$\iiint_E x \; dxdydz$$
$$= \iint_D \left(\int_0^{10-x-y} x \, dz \right) dxdy = \int_0^{10} \int_0^{10-x} \Big[xz \Big]_0^{10-x-y} dydx$$

$$=\int_0^{10}\int_0^{10-x} x(10-x-y)\,dy\,dx$$

$$=\int_0^{10}\int_0^{10-x} 10x-x^2-xy\,dy\,dx$$

$$=\int_0^{10}\left[10xy-x^2y-\frac{xy^2}{2}\right]_0^{10-x}dx$$

$$=\int_0^{10} 10x(10-x)-x^2(10-x)-\frac{x(10-x)^2}{2}\,dx$$

$$=\int_0^{10} 100x-10x^2-10x^2+x^3-\frac{x(100-2x+x^2)}{2}\,dx$$

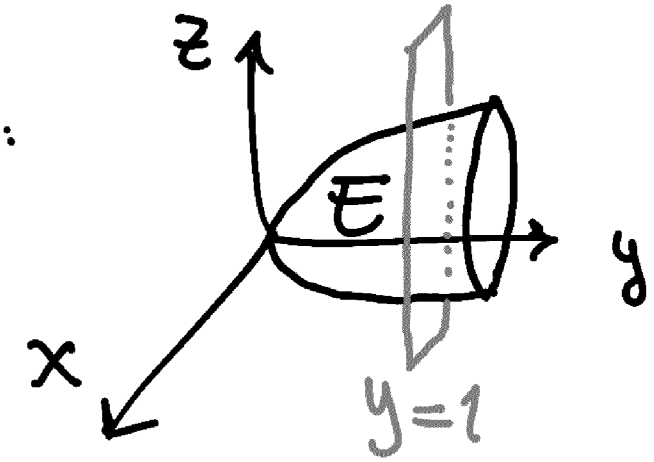
$$=\int_0^{10}\frac{x^3}{2}-10x^2+50x\,dx$$

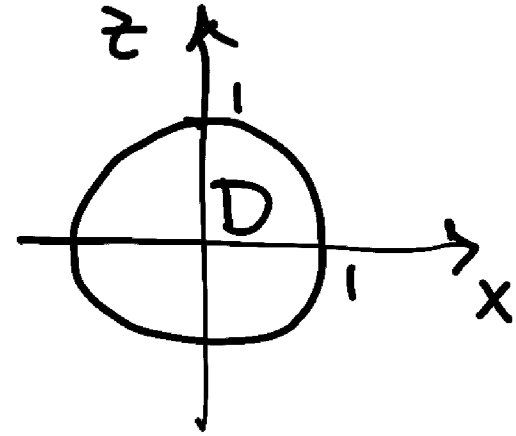
$$=\left[\frac{x^4}{8}-\frac{10x^2}{3}+25x^2\right]_0^{10}=\cdots$$

<u>Ex</u>: Compute $\iiint_E \sqrt{x^2+z^2}\,dx\,dy\,dz$

Where E is the region bounded by the paraboloid $y=x^2+z^2$ and the plane $y=1$.

Sol:





$(x,z) \in D =$ unit disk in xz-plane

$$x^2+z^2 \le y \le 1$$

$$\iiint_E \sqrt{x^2+z^2}\, dx\,dy\,dz = \iint_D \left(\int_{x^2+z^2}^{1} \sqrt{x^2+z^2}\, dy \right) dx\,dz$$

$$= \iint_D \sqrt{x^2+z^2}\,(1-(x^2+y^2))\,dx\,dz$$

Now change to polar coords:

$$D = \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$$

$$\iint_D \sqrt{x^2+z^2}\,(1-(x^2+y^2))\,dx\,dz = \int_0^{2\pi}\int_0^1 r(1-r^2)\,r\,dr\,d\theta$$

$$= 2\pi \int_0^1 r^2 - r^4 \, dr \, d\theta = 2\pi \left[\frac{r^3}{3} - \frac{r^5}{5} \right]_0^1$$

$$= 2\pi \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{4\pi}{15}.$$