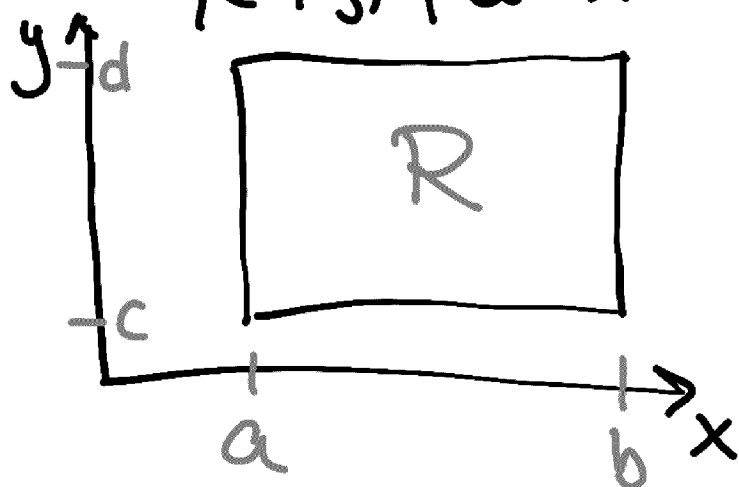


Recall: Double integrals:

$$\text{If } R = [a, b] \times [c, d]$$

$$= \{(x, y) \mid a \leq x \leq b \text{ and } c \leq y \leq d\},$$



$$\begin{aligned} \text{then } \iint_R f(x, y) dx dy &= \int_{y=c}^{y=d} \left(\int_{x=a}^{x=b} f(x, y) dx \right) dy \\ &= \int_{x=a}^{x=b} \left(\int_{y=c}^{y=d} f(x, y) dy \right) dx \end{aligned}$$

Ex: $\int_0^1 \int_1^2 \frac{x}{x^2 + y^2} dy dx$. Integrate in x

first. $\int_0^1 \frac{x}{x^2 + y^2} dx = \left[\begin{array}{l} u = x^2 + y^2 \\ du = 2x dx \\ u = 0 \Rightarrow x = y^2 \\ u = 1 \Rightarrow x = 1 + y^2 \end{array} \right]$

$$= \frac{1}{2} \int_{y^2}^{1+y^2} \frac{1}{u} du = \frac{1}{2} [\ln(u)]_{y^2}^{1+y^2}$$

$$= \frac{1}{2} (\ln(1+y^2) - \ln(y^2)) = \frac{1}{2} \ln\left(\frac{1+y^2}{y^2}\right)$$

$$= \frac{1}{2} \ln\left(\frac{1}{y^2} + 1\right). \text{ So}$$

$$\int_0^1 \int_1^2 \frac{x}{x^2+y^2} dy dx = \int_1^2 \left(\int_0^1 \frac{x}{x^2+y^2} dx \right) dy$$

$$= \int_1^2 \frac{1}{2} \underbrace{\ln\left(1 + \frac{1}{y^2}\right)}_u \underbrace{dy}_v. \text{ Partial integration:}$$

$$du = \frac{1}{1+\frac{1}{y^2}} \cdot \left(-\frac{2}{y^3}\right) dy, \text{ so we get}$$

$$\frac{1}{2} \left[\ln\left(1 + \frac{1}{y^2}\right) y \right]_1^2 - \frac{1}{2} \int_1^2 y \left(-\frac{2}{y^3}\right) \frac{1}{1+\frac{1}{y^2}} dy$$

$$= \frac{1}{2} (2 \ln(5/4) - \ln(2)) - \frac{1}{2} \int_1^2 \frac{-2}{y^2+1} dy$$

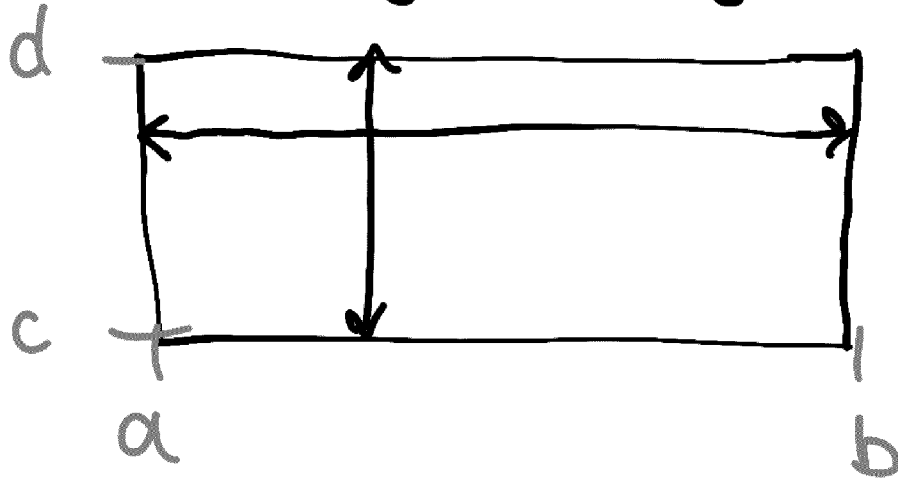
$$= \ln(5/4) - \ln(\sqrt{2}) + \int_1^2 \frac{1}{y^2+1} dy$$

$$= \ln\left(\frac{5}{4\sqrt{2}}\right) + [\arctan y]_1^2 = \ln\left(\frac{5}{4\sqrt{2}}\right)$$

$$+ \operatorname{arctan}(z) - \frac{\pi}{4}.$$

§5.2 More general regions

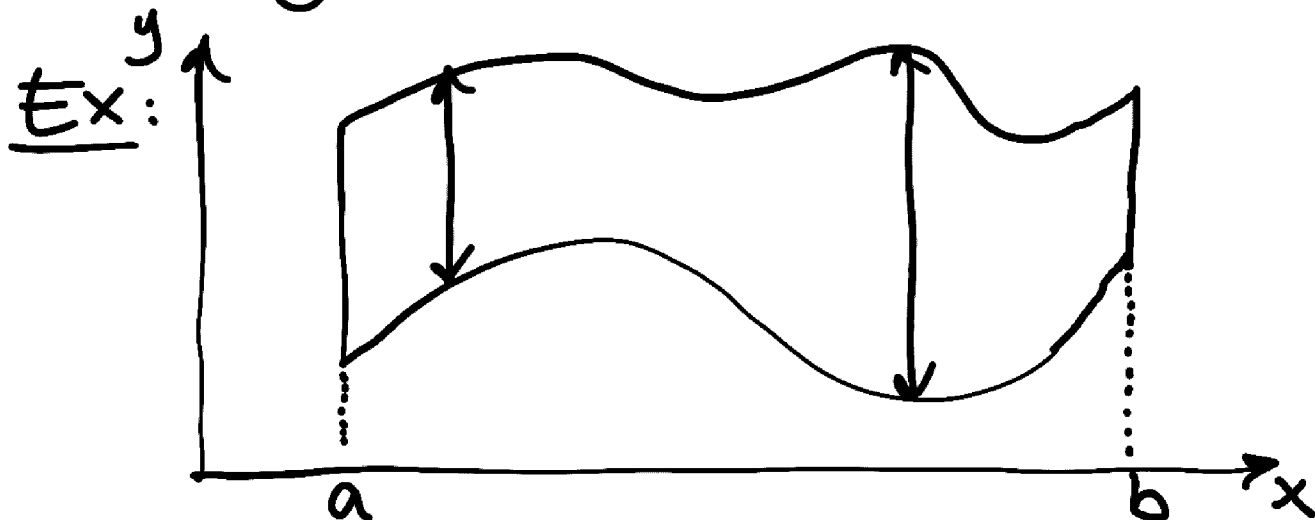
For rectangular regions:



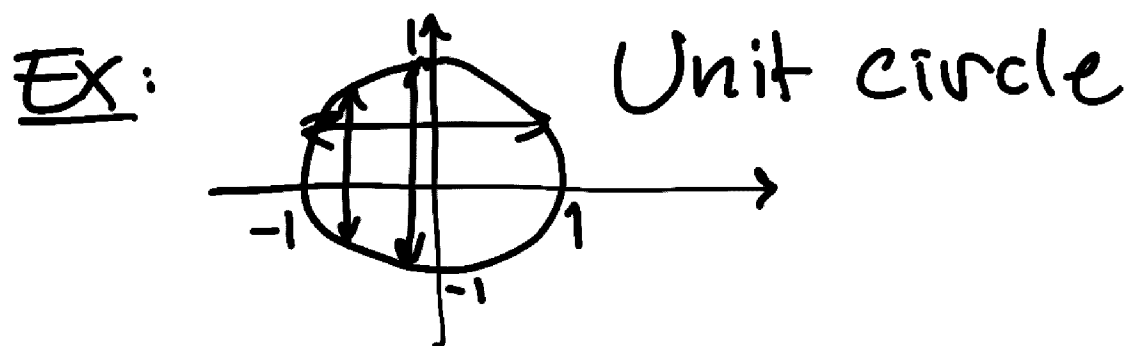
For every $y \in [c, d]$, The x integration bounds are $a \leq x \leq b$.

Similarly: for every $x \in [a, b]$, the y integration bounds are $c \leq y \leq d$.

More general regions:

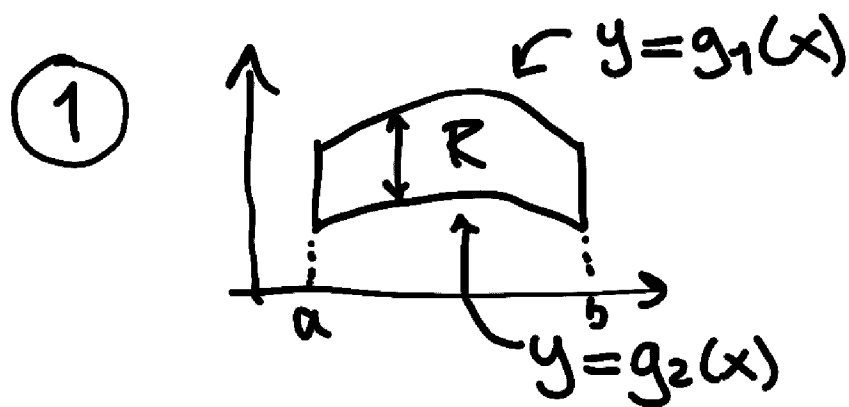


For each $x \in [a, b]$ the y integration bounds depend on x .



For every $y \in [-1, 1]$, the x integration bounds depend on x .

We focus on two kinds:

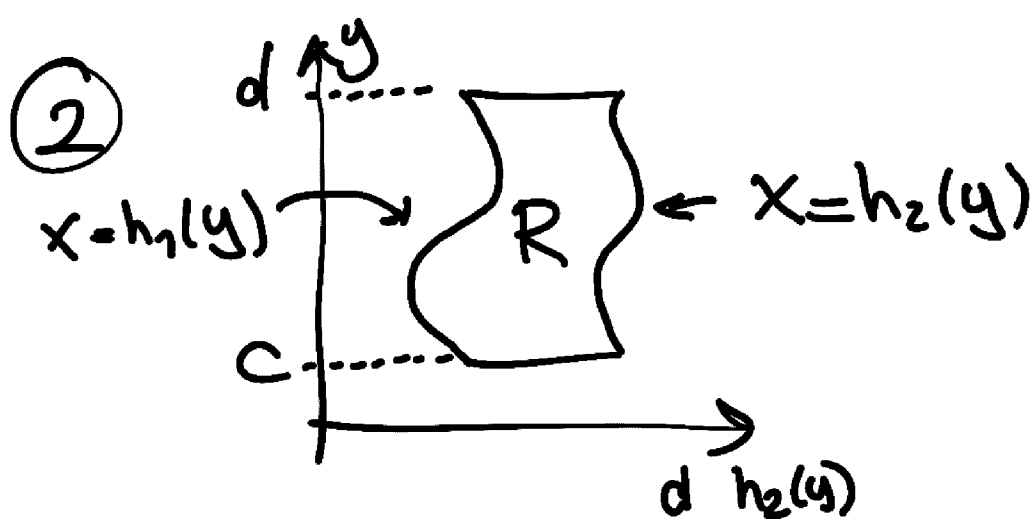


$$R = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}.$$

Then:

$$\iint_R f(x, y) dx dy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

⚠ You can not change integration order.



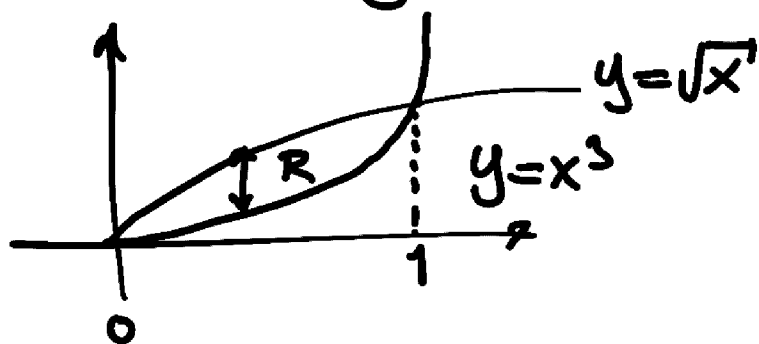
$$\iint_R f(x,y) dx dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y) dx dy$$

⚠ You can not change integration order.

Ex: Compute $\iint_R x y dx dy$, where

R is the region in the plane bounded by the curves $y = \sqrt{x}$ and $y = x^3$.

Sol:



$$g_1(x) = \sqrt{x}, \quad g_2(x) = x^3.$$

$$R = \{(x,y) \mid 0 \leq x \leq 1, x^3 \leq y \leq \sqrt{x}\}.$$

So

$$\begin{aligned}
\iint_R xy \, dx \, dy &= \int_0^1 \left(\int_{x^3}^{\sqrt{x}} 2xy \, dy \right) dx \\
&= \int_0^1 [xy^2]_{x^3}^{\sqrt{x}} dx = \int_0^1 (x \cdot 1x - x^7) dx \\
&= \int_0^1 x^2 - x^7 dx = \left[\frac{x^3}{3} - \frac{x^8}{8} \right] = \frac{1}{3} - \frac{1}{8} \\
&= \frac{8-3}{24} = \frac{5}{24}.
\end{aligned}$$

§5.3 Polar coords :

Recall that we described the area element dA in terms of dx and dy : $dA = dx \, dy$.

$$\iint_R f(x,y) \, dA = \iint_R f(x,y) \, dx \, dy.$$

Now let's change to polar coords:

$$\begin{cases} x = x(r, \theta) = r \cos \theta \\ y = y(r, \theta) = r \sin \theta \end{cases}$$

$$\begin{cases} dx = \cos \theta \, dr - r \sin \theta \, d\theta \\ dy = \sin \theta \, dr + r \cos \theta \, d\theta \end{cases}$$

$$\begin{aligned}
 dA = dx dy &= (\cos \theta dr - r \sin \theta d\theta)(\sin \theta dr + r \cos \theta d\theta) \\
 &= \sin \theta \cos \theta (dr)^2 + r \cos^2 \theta dr d\theta \\
 &\quad - r \sin^2 \theta d\theta dr - r^2 \sin \theta \cos \theta (d\theta)^2 \quad (*)
 \end{aligned}$$

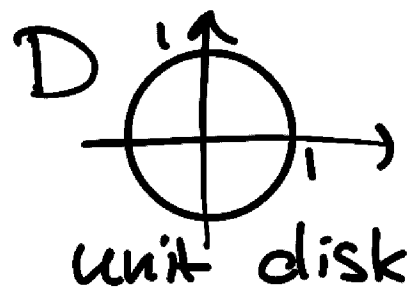
Two properties for "infinitesimal differentials": $\begin{cases} (dx)^2 = 0 \\ dx dy = -dy dx \end{cases}$

$$(*) = r(\cos^2 \theta + \sin^2 \theta) dr d\theta = r dr d\theta$$

$$\text{So } \iint_R f(x, y) dx dy = \iint_R f(r \cos \theta, r \sin \theta) r dr d\theta$$

Ex: Compute $\iint_D 4(x^2 + y^2) dx dy$ where

$$D = \{(x, y) \mid x^2 + y^2 \leq 1\}$$



This region is described as

$\{(r, \theta) \mid 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$ in polar coords

$$\begin{aligned}
 \iint_{\mathbb{R}^2} 4(x^2+y^2) dx dy &= \iint_{\mathbb{R}^2} 4(r^2) r dr d\theta \\
 &= \int_{\theta=0}^{2\pi} \int_{r=0}^1 4r^3 dr d\theta = \int_0^{2\pi} \left[r^4 \right]_0^1 d\theta \\
 &= \int_0^{2\pi} d\theta = 2\pi.
 \end{aligned}$$

Ex: Let's compute $\iint_{\mathbb{R}^2} e^{-x^2-y^2} dx dy$

Where $\mathbb{R}^2 = \mathbb{R}^2 =$ entire xy -plane
 $= (-\infty, \infty) \times (-\infty, \infty)$.

On the one hand

$$\begin{aligned}
 \iint_{\mathbb{R}^2} e^{-x^2-y^2} dx dy &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} \cdot e^{-y^2} dx dy \\
 &= \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right) \left(\int_{-\infty}^{\infty} e^{-y^2} dy \right) = \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right)^2 (*)
 \end{aligned}$$

But this integral is hard to compute

There is no easy primitive function to $f(x) = e^{-x^2}$.

Let's compute the original integral using polar coords instead.

R in polar coords:

$$R = \{(r, \theta) \mid 0 \leq r < \infty, 0 \leq \theta \leq 2\pi\}$$

$$\begin{aligned} \iint_R e^{-(x^2+y^2)} dx dy &= \int_0^\infty \int_0^{2\pi} e^{-r^2} r dr d\theta \\ &= 2\pi \int_0^\infty r e^{-r^2} dr = \left[\begin{array}{l} u = r^2 \\ du = 2r dr \\ r=0 \Rightarrow u=0 \\ r \rightarrow \infty \Rightarrow u \rightarrow \infty \end{array} \right] \end{aligned}$$

$$= \pi \int_0^\infty e^{-u} du = \pi \left[-e^{-u} \right]_0^\infty$$

$$= -\pi \left(\lim_{u \rightarrow \infty} \frac{1}{e^u} - 1 \right) = \pi.$$

Combine this with (*) to

$$\text{get } \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right)^2 = \pi \Leftrightarrow \boxed{\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}}$$