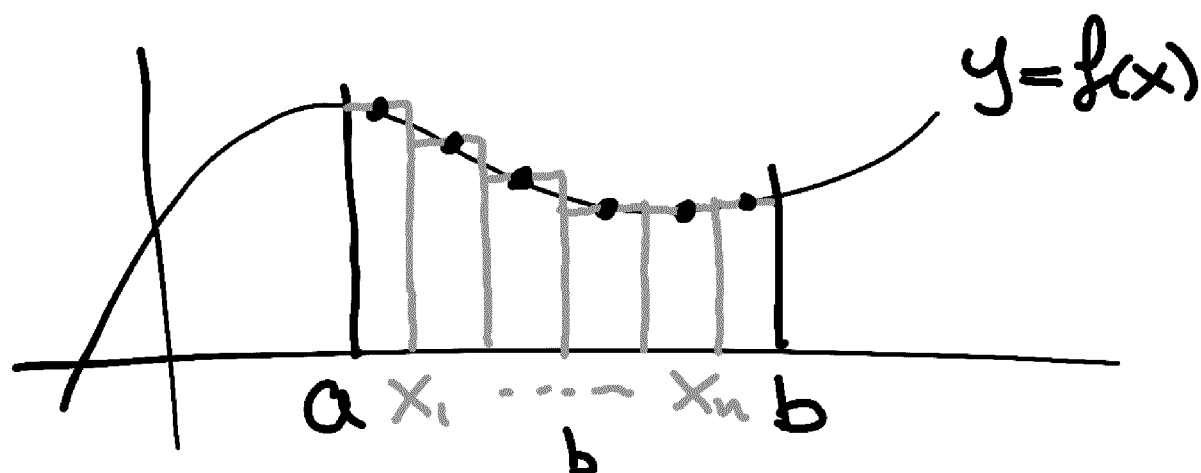


§5.1 Double integrals

Riemann sums:



To define $\int_a^b f(x) dx$, we divide $[a, b]$ into small intervals:

$$a = x_0 \leq x_1 \leq \dots \leq x_n \leq x_{n+1} = b$$

Pick a point $x_i^* \in [x_i, x_{i+1}]$
for $0 \leq i \leq n$

$$\text{Let } \Delta x_i = x_{i+1} - x_i$$

Area of each rectangle is

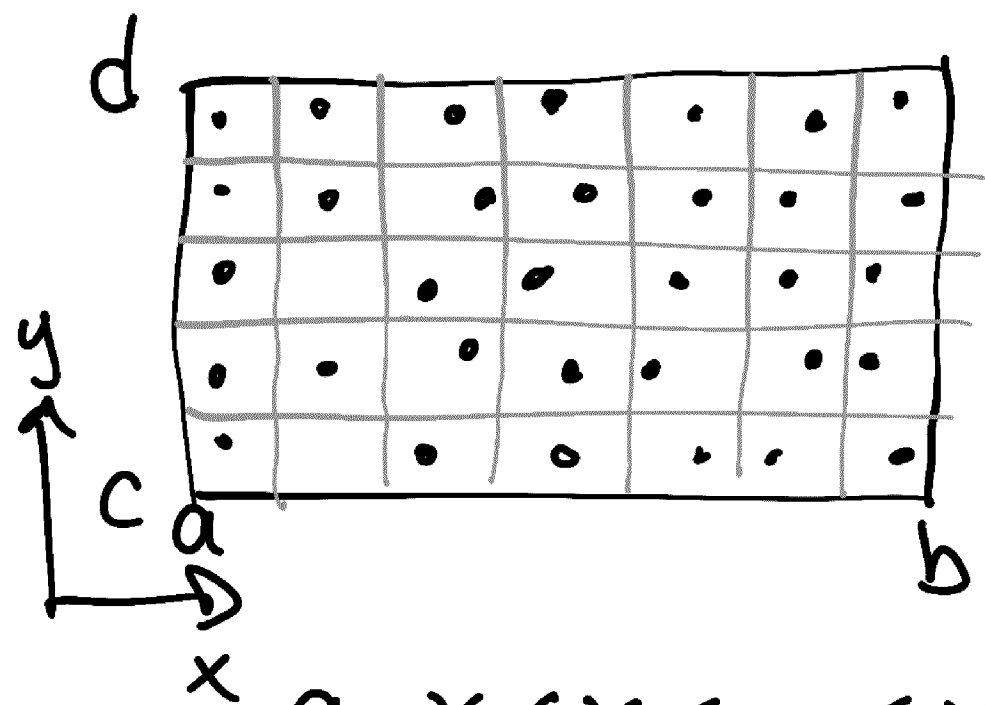
$f(x_i^*) \Delta x_i$. Now sum up

all contributions, and take the limit as the subdivision of the interval $[a, b]$ becomes finer & finer:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=0}^n f(x_i^*) \Delta x_i.$$

Now, we consider 2 variables:

$$[a, b] \times [c, d] = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$$



$$a = x_0 \leq x_1 \leq \dots \leq x_n \leq x_{n+1} = b$$

$$c = y_0 \leq y_1 \leq \dots \leq y_m \leq y_{m+1} = d$$

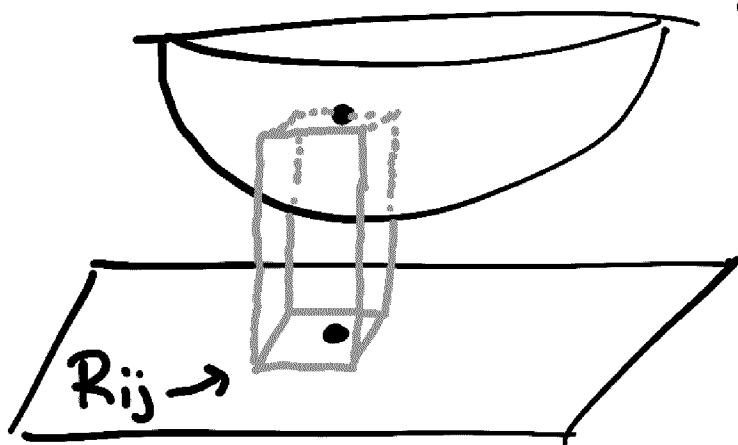
$$\Delta x_i = x_{i+1} - x_i, \quad \Delta y_i = y_{i+1} - y_i$$

Let R_{ij} be the rectangle

$$\{x_i \leq x \leq x_{i+1}, y_j \leq y \leq y_{j+1}\}$$

Let (x_{ij}^*, y_{ij}^*) be a point in R_{ij} .

$$z = f(x, y)$$



$$\Delta A_{ij} = \Delta x_i \Delta y_j$$

Volume: $f(x_{ij}^*, y_{ij}^*) \Delta A_{ij}$.

$R = [a, b] \times [c, d]$ integration region

$$\iint_R f(x, y) dA = \lim_{n, m \rightarrow \infty} \sum_{i=0}^n \sum_{j=0}^m f(x_{ij}^*, y_{ij}^*) \Delta A_{ij}$$

Sometimes we also write

$$\iint_R f(x, y) dx dy.$$

We always assume $f(x, y)$ is continuous when working with

integrals.

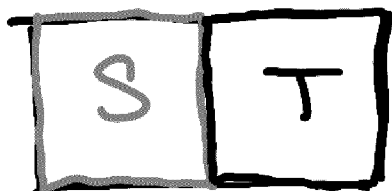
Properties:

$$\textcircled{1} \iint_R f(x,y) + g(x,y) dA = \iint_R f(x,y) dA + \iint_R g(x,y) dA$$

$\textcircled{2}$ If C is a constant

$$\iint_R C f(x,y) dA = C \iint_R f(x,y) dA$$

$\textcircled{3}$ Let $R = S \cup T$, and $S \cap T = \emptyset$

R 

$$\iint_R f(x,y) dA = \iint_S f(x,y) dA + \iint_T f(x,y) dA$$

$$\textcircled{4} \text{ If } f(x,y) \geq g(x,y), \iint_R f(x,y) dA \geq \iint_R g(x,y) dA$$

$\textcircled{5}$ If $m \leq f(x,y) \leq M$ then

$$m \cdot \text{Area}(R) \leq \iint_R f(x,y) dA \leq M \cdot \text{Area}(R)$$

⑥ In case $f(x,y) = g(x)h(y)$, and $R = [a,b] \times [c,d]$ then

$$\int\int_R f(x,y) dA = \left(\int_a^b g(x) dx \right) \left(\int_c^d h(y) dy \right).$$

How to compute?

If $R = [a,b] \times [c,d]$, then

$$\begin{aligned} \int\int_R f(x,y) dA &= \int_c^d \left[\int_a^b f(x,y) dx \right] dy \\ &= \int_a^b \left[\int_c^d f(x,y) dy \right] dx \end{aligned}$$

Ex: $f(x,y) = xy - 3xy^2$

$R = [0,2] \times [1,2]$, compute $\int\int_R f(x,y) dA$.

$$\int\int_R f(x,y) dA = \int_{y=1}^{y=2} \int_{x=0}^{x=2} xy - 3xy^2 dx dy$$

$$\begin{aligned}
&= \int_{y=1}^{y=2} \left(\left[\frac{x^2 y}{2} - \frac{3x^2 y^2}{2} \right]_{x=0}^{x=2} \right) dy \\
&= \int_1^2 2y - 6y^2 dy = \left[\frac{2y^2}{2} - \frac{6y^3}{3} \right]_1^2 \\
&= (4 - 16) - (1 - 2) = -11
\end{aligned}$$

Ex: Let $R = [0, \frac{\pi}{2}] \times [0, 1]$ & let's compute $\iint_R e^y \cos x \, dA$.

Since the integrand is $e^y \cos(x) = h(y)g(x)$,

$$\begin{aligned}
\iint_R e^y \cos x \, dA &= \left(\int_0^1 e^y dy \right) \left(\int_0^{\pi/2} \cos x \, dx \right) \\
&= [e^y]_0^1 [\sin x]_0^{\pi/2} = (e-1) \underbrace{(\sin \frac{\pi}{2})}_{=1} \\
&= e-1.
\end{aligned}$$

Ex: $\int_{y=-1}^0 \int_{x=0}^1 3x^2 + 2y \, dx \, dy.$

$$\begin{aligned} \textcircled{1} \quad & \int_{y=-1}^0 \left(\int_{x=0}^1 3x^2 + 2y \, dx \right) dy \\ &= \int_{y=-1}^0 [x^3 + 2xy]_{x=0}^{x=1} dy = \int_{-1}^0 1 + 2y \, dy \\ &= [y + y^2]_{-1}^0 = -(-1 + (-1)^2) = 0 \end{aligned}$$

Let's compute it in the other order!

$$\begin{aligned} \textcircled{2} \quad & \int_{x=0}^1 \left(\int_{y=-1}^0 3x^2 + 2y \, dy \right) dx \\ &= \int_{x=0}^1 [3x^2 y + y^2]_{y=-1}^{y=0} dx = \int_0^1 -(-3x^2 + 1) dx \\ &= [x^3 - x]_0^1 = 0 \end{aligned}$$

The order does not matter in general, as we mentioned above.

Ex: $R = [0, \pi] \times [1, 2]$. Compute

$$\iint_R x \sin(xy) dx dy.$$

$$\textcircled{1} \int_{y=1}^{y=2} \left(\int_{x=0}^{x=\pi} x \sin(xy) dx \right) dy.$$

Let's evaluate the inner integral first by integration by parts:

$$\int_{x=0}^{x=\pi} x \sin(xy) dx = \left[-\frac{x \cos(xy)}{y} \right]_{x=0}^{x=\pi} + \frac{1}{y} \int_0^{\pi} \cos(xy) dx$$

$$= -\frac{\pi \cos(\pi y)}{y} + \frac{1}{y} \left[\frac{\sin(xy)}{y} \right]_0^{\pi}$$

Now $\int_1^2 -\frac{\pi \cos(\pi y)}{y} dy$ is very hard to compute....

If we restart our computation & instead compute the integral with respect to y first, it's easier.

$$\textcircled{2} \int_{x=0}^{x=\pi} \left(\int_{y=1}^{y=2} x \sin(xy) dy \right) dx$$

$$= \int_{x=0}^{x=\pi} \left[\frac{-x \cos(xy)}{x} \right]_{y=1}^{y=2} dx$$

$$= \int_0^{\pi} -\cos(2x) + \cos(x) dx$$

$$= \left[-\frac{\sin(2x)}{2} + \sin(x) \right]_0^{\pi}$$

$$= \left(-\frac{\sin(2\pi)}{2} + \sin(\pi) \right) - \left(-\frac{\sin(0)}{2} + \sin(0) \right)$$

$$= 0$$

$$\text{Area}(R) = \iint_R dA \quad (\text{integrand is } f(x,y)=1)$$

The average value of $f(x,y)$ in the region R is

$$f_{\text{ave}} = \frac{1}{\text{Area}(R)} \iint_R f(x,y) dA = \frac{\iint_R f(x,y) dA}{\iint_R dA}$$

$$\begin{aligned}
 \text{Ex: } & \int_{x=1}^e \int_{y=1}^2 \frac{\ln(x)}{x} dy dx \\
 &= \left(\int_1^e \frac{\ln(x)}{x} dx \right) \left(\int_1^2 dy \right) \\
 &= \int_1^e \frac{\ln(x)}{x} dx = \left[\begin{array}{l} u = \ln(x) \\ du = \frac{1}{x} dx \\ x=1 \Rightarrow u=0 \\ x=e \Rightarrow u=1 \end{array} \right] = \int_0^1 u du \\
 &= \left[\frac{u^2}{2} \right]_0^1 = \frac{1}{2}.
 \end{aligned}$$

$$R = [1, e] \times [1, 2]$$

Average value of $f(x, y) = \frac{\ln(x)}{x}$

in R is

$$\begin{aligned}
 f_{\text{ave}} &= \frac{1}{\text{Area}(R)} \iint_R \frac{\ln(x)}{x} dx \\
 &= \frac{1}{(e-1)(2-1)} = \frac{1}{2(e-1)}.
 \end{aligned}$$
