

Recall: • (x_0, y_0) critical point of f if $\nabla f(x_0, y_0) = 0$.

• "Second derivative test":

$$D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2$$

↖ discriminant

Let (x_0, y_0) be a critical point.

- $D(x_0, y_0) > 0$, $f_{xx}(x_0, y_0) > 0$
 $\Rightarrow$ (x_0, y_0) is a local min
- $D(x_0, y_0) > 0$, $f_{xx}(x_0, y_0) < 0$
 $\Rightarrow$ (x_0, y_0) is a local max
- $D(x_0, y_0) < 0$
 $\Rightarrow$ (x_0, y_0) Saddle
- $D(x_0, y_0) = 0$
 $\Rightarrow$ Inconclusive
- If $f = f(x, y)$ is defined on the domain $D \subset \mathbb{R}^2$,

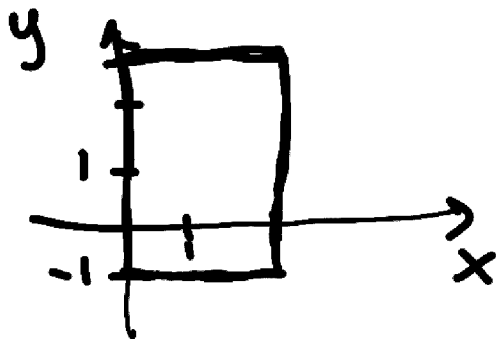
then we find global max/min by:

1. Find critical pts (x_0, y_0)
 2. Compute values $f(x_0, y_0)$
 3. Find min/max along the boundary $\partial D \subset \mathbb{R}^2$
 4. Compare all values.
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Ex: Find global max/min of

$$f(x, y) = 4x^2 - 2xy + 6y^2 - 8x + 2y + 3$$

in the domain $\{(x, y) \mid 0 \leq x \leq 2, -1 \leq y \leq 3\}$



1. Find critical points

$$\begin{aligned}\nabla f(x, y) &= \langle 8x - 2y - 8, -2x + 12y + 2 \rangle \\ &= \langle 0, 0 \rangle\end{aligned}$$

$$\begin{cases} 8x - 2y - 8 = 0 & \textcircled{1} \\ -2x + 12y + 2 = 0 & \textcircled{2} \end{cases}$$

$$\text{Eq ①} \Leftrightarrow 2y = 8x - 8 \Leftrightarrow y = 4x - 4$$

Substitute into eq ②:

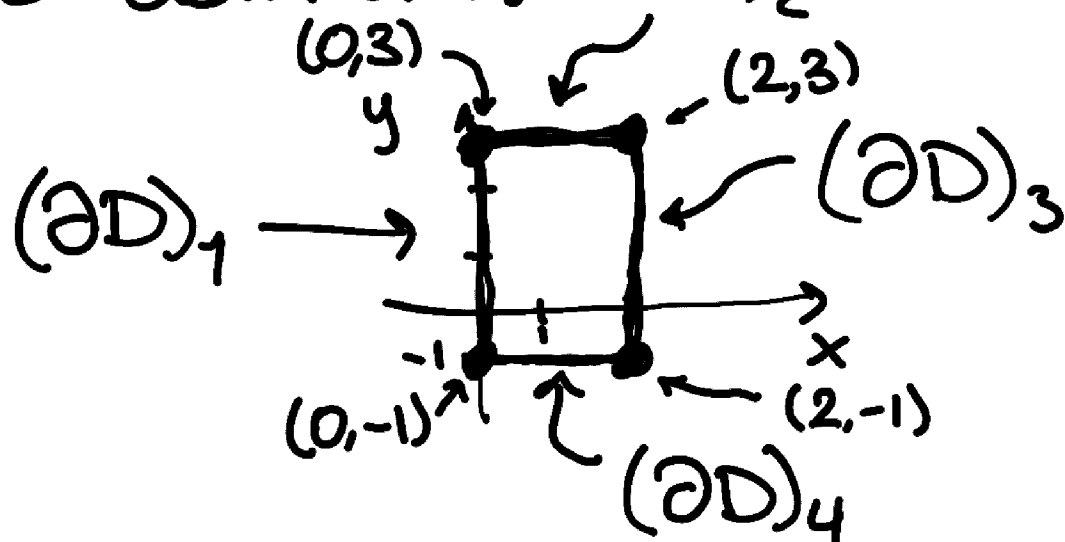
$$-2x + 12(4x - 4) + 2 = 0$$

$$\Leftrightarrow -2x + 48x - 48 + 2 = 0$$

$$\Leftrightarrow 46x - 46 = 0 \Leftrightarrow \boxed{x=1} \\ \Rightarrow \boxed{y=0}$$

$$f(1,0) = 4 - 8 + 3 = \boxed{-1}$$

Now consider the boundary of the domain: $(\partial D)_2$



$$(\partial D)_1 = \{x=0, -1 \leq y \leq 3\}$$

$$(\partial D)_2 = \{0 \leq x \leq 2, y=3\}$$

$$(\partial D)_3 = \{x=2, -1 \leq y \leq 3\}$$

$$(\partial D)_4 = \{0 \leq x \leq 2, y = -1\}$$

We check each boundary separately, and find min/max of the function along each one.

$$(\partial D)_1: f(y) = 6y^2 + 2y + 3, -1 \leq y \leq 3$$

$$f'(y) = 12y + 2 = 0 \Leftrightarrow y = -\frac{1}{6}$$

$$\begin{aligned} f\left(-\frac{1}{6}\right) &= 6\left(-\frac{1}{6}\right)^2 + 2\left(-\frac{1}{6}\right) + 3 \\ &= \frac{1}{6} - \frac{2}{6} + 3 = 3 - \frac{1}{6} = \boxed{\frac{17}{6}} \end{aligned}$$

$$\begin{aligned} (\partial D)_2: f(x) &= 4x^2 - 6x + 54 - 8x + 6 + 3 \\ &= 4x^2 - 14x + 63 = 0 \\ &\quad 0 \leq x \leq 2 \end{aligned}$$

$$f'(x) = 8x - 14 = 0 \Leftrightarrow x = \frac{14}{8} = \frac{7}{4}$$

$$\begin{aligned} f\left(\frac{7}{4}\right) &= 4 \cdot \frac{7^2}{4^2} - 14 \cdot \frac{7}{4} + 63 \\ &= \frac{49}{4} - \frac{49 \cdot 2}{4} + 63 = \boxed{-\frac{49}{4} + 63} \end{aligned}$$

$$\begin{aligned} (\partial D)_3: f(y) &= 16 - 4y + 6y^3 - 16 + 2y + 3 \\ &= 6y^2 - 2y + 3 \quad -1 \leq y \leq 3 \end{aligned}$$

$$f'(y) = 12y - 2 = 0 \Leftrightarrow y = \frac{1}{6}$$

$$\begin{aligned} f\left(\frac{1}{6}\right) &= 6 \cdot \frac{1}{6^2} - \frac{2}{6} + 3 = \frac{1}{6} - \frac{2}{6} + 3 \\ &= 3 - \frac{1}{6} = \boxed{\frac{17}{6}} \end{aligned}$$

$$\begin{aligned} (\partial D)_4: f(x) &= 4x^2 + 2x + 6 - 8x - 2 + 3 \\ &= 4x^2 - 6x + 7 \quad 0 \leq x \leq 2 \end{aligned}$$

$$f'(x) = 8x - 6 = 0 \Leftrightarrow x = \frac{6}{8} = \frac{3}{4}$$

$$\begin{aligned} f\left(\frac{3}{4}\right) &= 4 \cdot \frac{9}{4^2} - \frac{18}{4} + 7 = \frac{36}{4} - \frac{18}{4} + 7 \\ &= \frac{18}{4} + 7 = \boxed{\frac{46}{4}} \end{aligned}$$

Corner points:

$$f(0, -1) = 6 - 2 + 3 = \boxed{7}$$

$$f(2, -1) = 16 + 4 + 6 - 16 - 2 + 3 = \boxed{11}$$

$$f(0, 3) = 54 + 6 + 3 = \boxed{63}$$

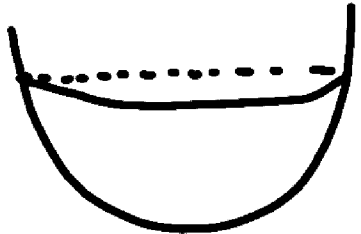
$$f(2, 3) = 16 - 18 + 54 - 16 + 6 + 3 = \boxed{45}$$

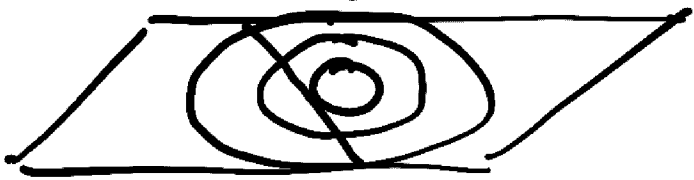
Comparing all values we find
global min at $(x, y) = (1, 0)$
with value -1

global max at $(x,y)=(0,3)$
with value 63

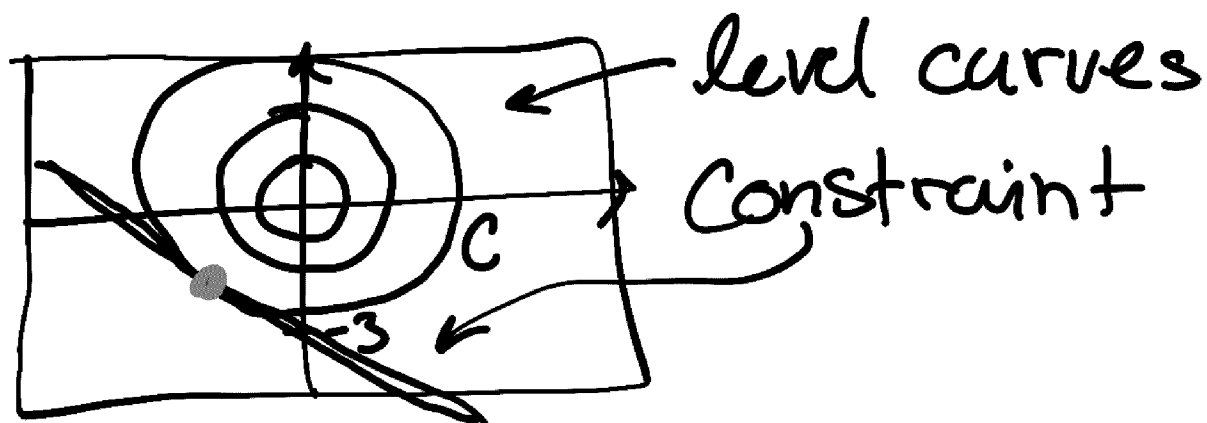
§4.8 Lagrange multipliers

The method of Lagrange multipliers is used to maximize/minimize a function with respect to one or more constraints.

Ex:  $f(x,y) = x^2 + y^2$



Suppose we want to minimize $f(x,y)$ with respect to the constraint $x+y+3=0$.



The minimum is the value of the level curve that is tangent to the constraint.

$$f(x,y) = x^2 + y^2, \nabla f = \langle 2x, 2y \rangle$$

$$g(x,y) = x + y + 3, \nabla g = \langle 1, 1 \rangle$$

∇f normal to level curves

∇g normal to the constraint.

Constraint is tangent to level curves if $\nabla f = \lambda \nabla g$ for some λ .

Theorem: Let $f = f(x,y)$, $g = g(x,y)$.

Suppose (x_0, y_0) is a local max/min of $f(x,y)$ when restricted to the curve $g(x,y) = 0$, and $\nabla g(x_0, y_0) \neq 0$.

Then there is some scalar λ ("Lagrange multiplier") such that

$$\boxed{\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)}$$

How to use it:

If we want to optimize $f(x, y)$ with respect to the constraint $g(x, y) = 0$ we solve

$$\begin{cases} \nabla f(x, y) = \lambda \nabla g(x, y) \\ g(x, y) = 0 \end{cases}$$

All points (x, y) that solve this are candidates for min/max.

Ex. $f(x, y) = x^2 + 4y^2 - 2x + 8y$
 $g(x, y) = x + 2y - 7.$

$$\nabla f(x, y) = \langle 2x - 2, 8y + 8 \rangle$$

$$\nabla g(x, y) = \langle 1, 2 \rangle$$

$$\begin{cases} \langle 2x - 2, 8y + 8 \rangle = \lambda \langle 1, 2 \rangle \\ x + 2y - 7 = 0 \end{cases}$$

$$\begin{cases} 2x-2=\lambda & \textcircled{1} \\ 8y+8=2\lambda & \textcircled{2} \\ x+2y-7=0 & \textcircled{3} \end{cases}$$

Insert $\textcircled{1}$ in $\textcircled{2}$: $\begin{cases} 8y+8=2(2x-2) & \textcircled{1'} \\ x=7-2y & \textcircled{2'} \end{cases}$

Insert $\textcircled{2'}$ in $\textcircled{1'}$

$$8y+8=2(2(7-2y)-2)$$

$$\Leftrightarrow 8y+8=2(14-4y-2)=28-8y-4$$

$$\Leftrightarrow 16y = 16 \Leftrightarrow \boxed{y=1}$$

$$\Rightarrow x=7-2 \cdot 1=5$$

Only solution is $(x,y)=(5,1)$.

$$f(5,1)=25+4-10+8=27$$

To figure out if it's a max or min, find some other point on the line $g(x,y)=0 \Leftrightarrow x+2y=7$ &

Compare the values of f . For example

$$x=6 \quad \text{gives } y=\frac{7-6}{2}=\frac{1}{2}$$

$$f(6, \frac{1}{2}) = 36 + 4 \cdot \frac{1}{2} - 12 + \frac{8}{2} = 24 + 1 + 4 = 29$$

> 27

So $(x, y) = (5, 1)$ with $f(5, 1) = 27$
appears to be a minimum.
