

Recall: $f = f(x, y)$

- $\vec{v} = \langle v_x, v_y \rangle$. Directional derivative of f in the direction of $\vec{v}$ is

$$D_{\vec{v}} f = \lim_{h \rightarrow 0} \frac{f(x + h v_x, y + h v_y) - f(x, y)}{h \|\vec{v}\|}$$

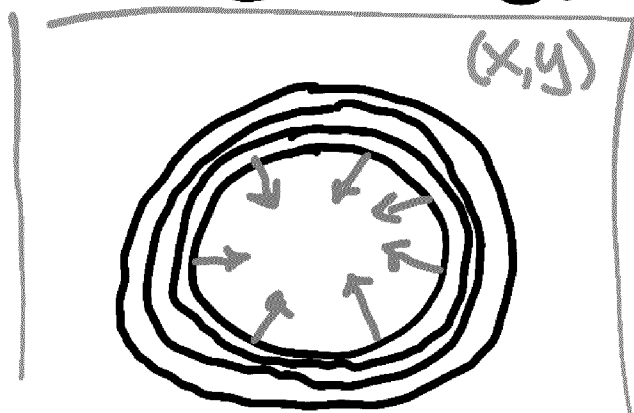
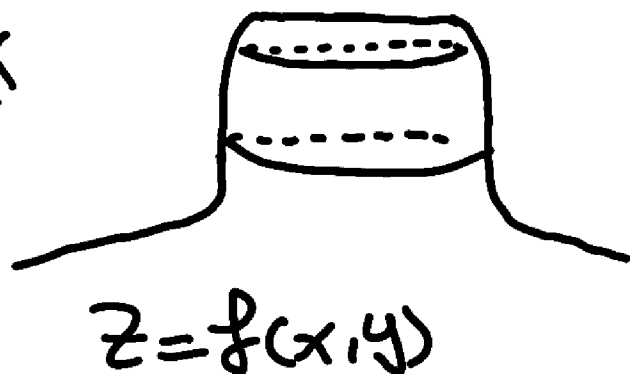
- The gradient of f is

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$$

- $D_{\vec{v}} f = \nabla f(x, y) \cdot \frac{\vec{v}}{\|\vec{v}\|}$

The gradient $\nabla f(x, y)$ is orthogonal to level curves.

Ex



§4.7 Optimization problems

Def: A point (x_0, y_0) is called a critical point for $f = f(x, y)$ if

- (1) $\nabla f(x_0, y_0) = \vec{0}$, or
- (2) $\nabla f(x_0, y_0)$ doesn't exist.

Ex: $f(x, y) = x^3 + 2xy - 2x - 4y$

Let's find all critical points.

$$\nabla f(x, y) = \langle 3x^2 + 2y - 2, 2x - 4 \rangle = \langle 0, 0 \rangle$$

$$\begin{cases} 3x^2 + 2y - 2 = 0 \\ 2x - 4 = 0 \Rightarrow \boxed{x = 2} \end{cases}$$

Substitute into first eq.:

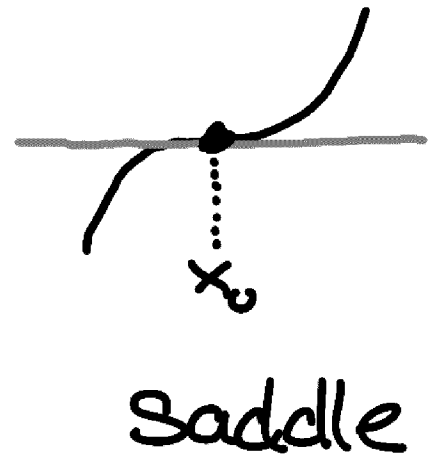
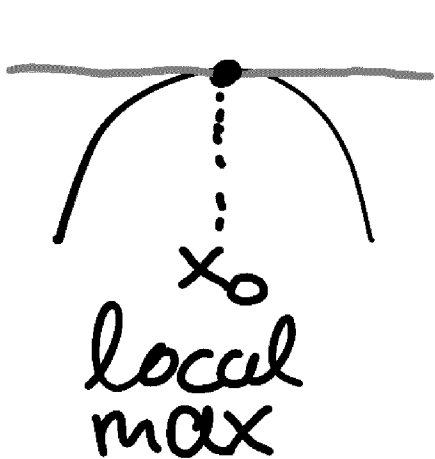
$$3 \cdot 2^2 + 2y - 2 = 0 \Rightarrow \boxed{y = -5}$$

The only critical point is
 $(x, y) = (2, -5)$.

Intuitively, what is a critical point?

In single variable calculus:

If $f'(x_0) = 0$ then x_0 is a critical point. This means that the slope $= 0$.



In our case: $\nabla f(x_0, y_0) = \vec{0}$

means
$$\begin{cases} f_x(x_0, y_0) = 0 \\ f_y(x_0, y_0) = 0 \end{cases}$$

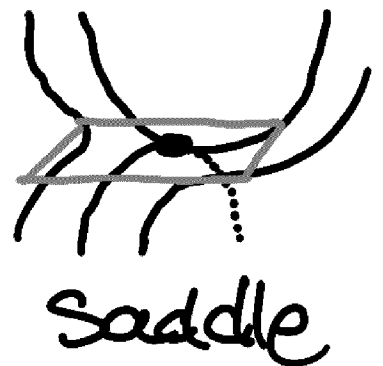
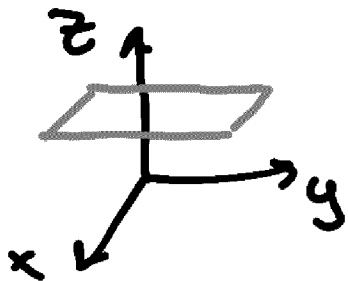
So if f has a tangent plane at $(x, y) = (x_0, y_0)$,

its given by

$$\underbrace{f_x(x_0, y_0)}_{=0}(x-x_0) + \underbrace{f_y(x_0, y_0)}_{=0}(y-y_0) + (z - f(x_0, y_0)) = 0$$

$$\Leftrightarrow z = f(x_0, y_0).$$

This is a horizontal plane



Def: Let $f(x, y)$ be continuous.

(1)

We say (x_0, y_0) is a local maximum

if $f(x_0, y_0) \geq f(x, y)$ for all (x, y) close to (x_0, y_0)

(2)

We say (x_0, y_0) is a local minimum if $f(x_0, y_0) \leq f(x, y)$ for all (x, y) near (x_0, y_0) .

(3) We say (x_0, y_0) is a saddle point if $\nabla f(x_0, y_0) = \vec{0}$ and (x_0, y_0) is neither a local max, nor a local min.

For functions of one variable, we can conveniently find min/max/saddles using $f''(x)$.

$$f''(x_0) < 0 \quad f''(x_0) > 0$$



Theorem (Second derivative test)

Let f be differentiable. Suppose

$$\nabla f(x_0, y_0) = \vec{0}. \text{ Define}$$

$$D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$$

"discriminant"

(1) If $D > 0$ and $f_{xx}(x_0, y_0) > 0$

then (x_0, y_0) is a local min

(2) If $D > 0$ and $f_{xx}(x_0, y_0) < 0$

then (x_0, y_0) is a local max

(3) If $D < 0$, (x_0, y_0) is a

Saddle

(4) If $D = 0$ the test is

inconclusive.

Ex: $f(x, y) = 4x^2 + 9y^2 + 8x - 36y + 24$

Let's find and classify all

Critical points.

$$\nabla f(x,y) = \langle 8x+8, 18y-36 \rangle = \vec{0}$$

$$\Rightarrow (x,y) = (-1,2)$$

$$f_{xx}(x,y) = 8, \quad f_{yy}(x,y) = 18$$

$$f_{xy}(x,y) = 0.$$

$$\begin{aligned} D &= f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2 \\ &= 8 \cdot 18 = 144 \end{aligned}$$

So $D > 0$ and $f_{xx}(-1,2) > 0$

So $(x,y) = (-1,2)$ is a local min

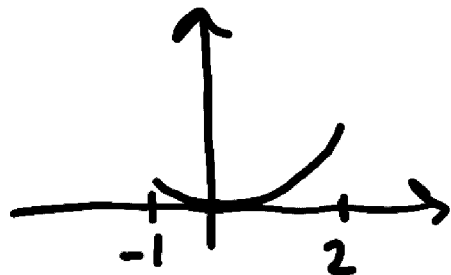
If (x_0, y_0) is a critical point,
 $f(x_0, y_0)$ is the critical value.

The global min/max (if it exists) is the min/max value of $f(x,y)$.

Global min/max can occur at local min/max, or at the boundary of the domain of definition.

Ex: One variable:

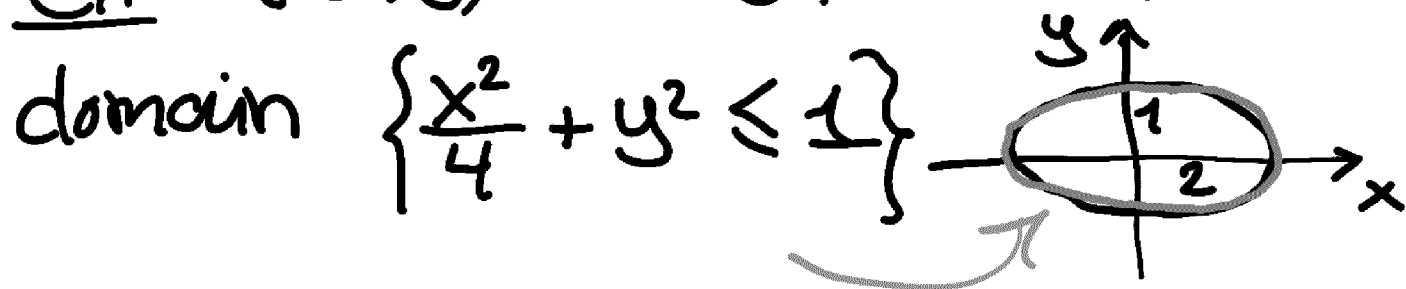
$$f(x) = x^2, \quad x \in [-1, 2].$$

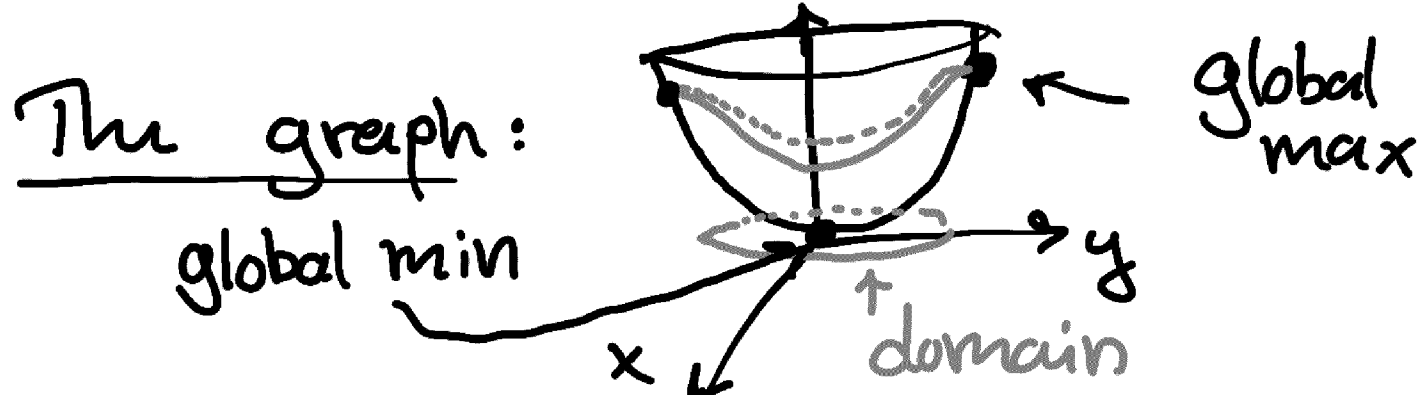


- global min is $f(0) = 0$
Occurring at $x = 0$ (local min)
 - global max on $-1 \leq x \leq 2$ is $f(2) = 4$, at the boundary point $\underline{x = 2}$.
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Ex: $f(x, y) = x^2 + y^2$, on the

domain $\left\{ \frac{x^2}{4} + y^2 \leq 1 \right\}$





Intuitively, the global min is $f(0,0)=0$, at the local min.

The global max can be "seen" from the graph to lie along the boundary, at $(x,y)=(\pm 2,0)$
 $f(\pm 2,0)=4$.

Let's verify all of this via computation.

$$\nabla f(x,y) = \langle 2x, 2y \rangle = \langle 0, 0 \rangle$$

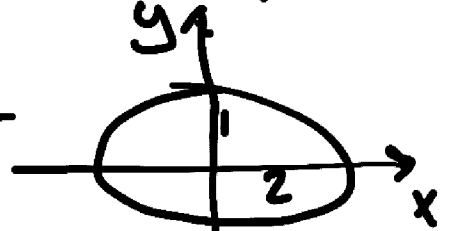
$$(x,y) = (0,0)$$

$$\begin{aligned} D &= f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2 \\ &= 2 \cdot 2 - 0 = 4 > 0 \end{aligned}$$

$$f_{xx}(0,0) = 2 > 0$$

$\Rightarrow (x,y) = (0,0)$ local min
by the second derivative test.

Now, let's consider the
boundary of the domain,
which is $\frac{x^2}{4} + y^2 = 1$



$$x^2 + 4y^2 = 4 \Leftrightarrow x^2 = 4 - 4y^2 \quad (*)$$

The function along this
boundary is therefore given
by: $f(x,y) = x^2 + y^2 = 4 - 4y^2 + y^2$
 $= 4 - 3y^2$.

We now need to maximize
this function of one variable!

$$g(y) = 4 - 3y^2. \quad -1 \leq y \leq 1$$

To do this, we can use our Calc I knowledge.

$$g'(y) = -6y = 0$$

$$y=0, \quad g'(y) = -6 < 0$$

so $y=0$ is a local max
with value $g(0)=4$.

Boundary points: $g(-1)=g(1)=1$

$$y=0 \xrightarrow{(*)} x^2=4 \Rightarrow \boxed{x=\pm 2}$$

so $(\pm 2, 0)$ are points realizing
the global max of f , at

$$\left\{ \frac{x^2}{4} + y^2 \leq 1 \right\}. \quad f(\pm 2, 0) = 4 \text{ max}$$
$$f(0, 0) = 0 \text{ min}$$
