MAT 203 L11 Tue Oct 7

Recall: • Partial derivatives

$$\frac{\partial f}{\partial x} = f_x = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$\frac{\partial f}{\partial y} = f_y = \lim_{k \to 0} \frac{f(x,y+k) - f(x,y)}{k}$$

• Let S be the graph, $z = f(x,y)$ if S has a tangent plane at $(x,y) = (a,b)$ it is given by

$$f_x(a,b)(x-a) + f_y(a,b)(y-b) - (z - f(a,b)) = 0$$

When does $z = f(x,y)$ have tangent planes? The answer is: when f is <u>differentiable</u>

<u>Def</u>: f is differentiable at $(x,y) = (a,b)$ if when writing

$$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$+ E(x,y) \leftarrow$ error term

we have

$$\lim_{(x,y)\to(a,b)} \frac{E(x,y)}{\sqrt{(x-a)^2+(y-b)^2}} = 0$$

<u>Ex</u>:

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) - (0,0) \end{cases}$$

Is f differentiable at the origin?

First let's compute $f_x(0,0)$, $f_y(0,0)$

Since f is defined piecewise, we need to use the definition.

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0}{h} = 0$$

$f_y(0,0) = $ [similar limit] $= 0$.

Now, let's find the error term as in the def of differentiability

$$f(x,y) = \underbrace{f(0,0)}_{=0} + \underbrace{f_x(0,0)}_{=0}x + \underbrace{f_y(0,0)}_{=0}y + E(x,y)$$

$$= E(x,y) = \frac{xy}{\sqrt{x^2+y^2}}$$

$$\lim_{(x,y)\to(0,0)} \frac{E(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

and this limit doesn't exist, b/c along the line $y=0$ it is

$$\boxed{y=0} \quad \lim_{(x,y)\to(0,0)} \frac{0}{x^2} = 0$$

Along the line $x=y$ it is

$$\boxed{x=y} \quad \lim_{x\to 0} \frac{x^2}{x^2+x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

Since the limit doesn't exist, $f(x,y)$ is not differentiable at $(x,y)=(0,0)$. But note, that the partial derivatives <u>do</u> exist!

<u>Remark</u>: If f is differentiable, its partial derivatives exist, but <u>not the other way</u> <u>around</u>.

<u>Ex</u>: $f(x,y)=xe^{xy}$. Is differentiable at $(x,y)=(0,0)$?

$$f_x(x,y)=e^{xy}+xye^{xy}=(1+xy)e^{xy}$$
$$f_y(x,y)=x^2e^{xy}$$

$$f(1,0)=0\cdot e^0=0$$
$$f_x(1,0)=(1+0)e^0=1$$
$$f_y(1,0)=0e^0=0$$

$$f(x,y)=\underbrace{f(0,0)}_{=0}+\underbrace{f_x(0,0)}_{=1}x+\underbrace{f_y(0,0)}_{=0}y+E(x,y)$$

$$\Leftrightarrow E(x,y)=f(x,y)-x=xe^{xy}-x$$

$$\lim_{(x,y)\to(0,0)} \frac{xe^{xy}-x}{\sqrt{x^2+y^2}}.$$ Does it exist?

Let's try to check the limit along lines $y=kx$.

$$\lim_{x\to 0} \frac{xe^{kx^2}-x}{\sqrt{x^2+k^2x^2}} = \lim_{x\to 0} \frac{xe^{kx^2}-x}{\sqrt{1+k^2}\,|x|}$$

$$= \lim_{x\to 0} \frac{\operatorname{sgn}(x)}{\sqrt{1+k^2}} \underbrace{\left(e^{kx^2}-1\right)}_{\to 0} = 0$$

We guess that the limit exists, and now we prove it:

Remember $\lim_{(x,y)\to(a,b)} f(x,y) = L$

if $|f(x,y)-L|$ can be arbitrarily small by making $\text{dist}((x,y),(a,b))$ arbitrarily small.

We have $x \le \sqrt{x^2+y^2}$ for x, y very close to 0. So

$$\left|\frac{x(e^{xy}-1)}{\sqrt{x^2+y^2}}\right| \leq \left|\frac{\sqrt{x^2+y^2}\,(e^{xy}-1)}{\sqrt{x^2+y^2}}\right|$$

$$= |e^{xy}-1| \longrightarrow 0,$$ which

shows
$$\lim_{(x,y)\to(0,0)} \frac{xe^{xy}-x}{\sqrt{x^2+y^2}} = 0.$$

<u>Ex</u>: $f(x,y) = x^2+3y$. Is f differentiable at every point?

$f_x(x,y) = 2x, \quad f_y(x,y) = 3.$

$$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + E(x,y)$$

$$\Leftrightarrow E(x,y) = f(x,y) - f(a,b) - 2a(x-a) - 3(y-b)$$

$$= x^2 \cancel{+3y} - a^2 \cancel{-3b} - 2ax + 2a^2 \cancel{-3y} \cancel{+3b}$$

$$= x^2 + a^2 - 2ax = (x-a)^2.$$ Now

$$\left|\frac{(x-a)^2}{\sqrt{(x-a)^2+(y-b)^2}}\right| \leq \left|\frac{(x-a)^2}{\sqrt{(x-a)^2}}\right| = |x-a|$$

$\longrightarrow 0$ as $(x,y) \to (a,b)$, so $f(x,y)$ is differentiable at every pt.

Thm: If f is differentiable, f is continuous.

§4.5 Chain rule

Chain rule in a single variable:

$$\boxed{\begin{aligned}\frac{d}{dx} f(g(x)) &= \frac{df}{dg} \cdot \frac{dg}{dx} \\ &= f'(g(x)) \cdot g'(x).\end{aligned}}$$

Let's first consider $f(x,y)$ where $x = x(t)$, $y = y(t)$ depends on another variable. Then:

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}}$$

$f(\ x(t),\ \ y(t)\ )$

$x(t) \qquad\qquad\qquad\qquad y(t)$

$t \qquad\qquad\qquad\qquad\qquad t$

Ex: $f(x,y) = 4x^2 + 3y^2$

$$x = x(t) = \sin t$$
$$y = y(t) = \cos t$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= (8x)(\cos t) + (6y)(-\sin t)$$

$$= 8\sin t\cos t - 6\cos t\sin t$$

$$= 2\sin t\cos t = \sin(2t)$$

Ex: $f(x,y) = \sqrt{x^2 - y^2}$

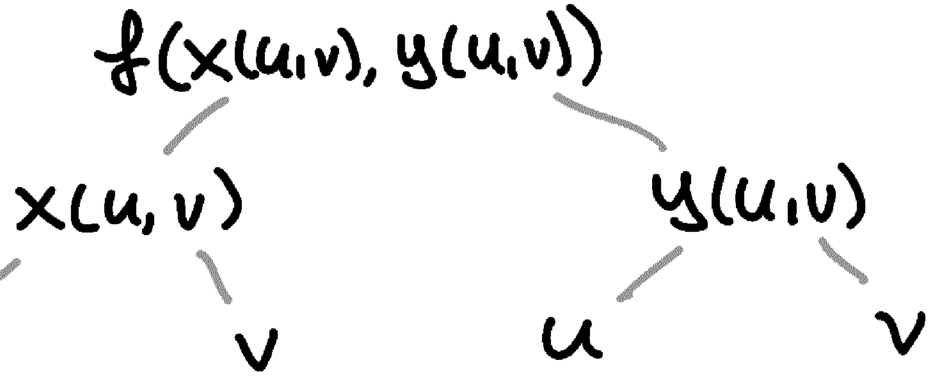
$$x = x(t) = e^{2t}$$
$$y = y(t) = e^{-t}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= \left(\frac{2x}{\sqrt{x^2-y^2}}\right)\left(2e^{2t}\right) + \left(\frac{-2y}{\sqrt{x^2-y^2}}\right)\left(-e^{-t}\right)$$

$$= \frac{4e^{4t^2}+2e^{t^2}}{\sqrt{e^{4t^2}-e^{t^2}}}$$

Now consider $f(x,y)$, and let $x = x(u,v)$, $y = y(u,v)$. The function f depends on u, v, $f = f(x(u,v), y(u,v))$.



$$\boxed{\begin{aligned} \frac{\partial f}{\partial u} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \\ \frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} \end{aligned}}$$

Ex: $f(x,y) = e^{xy}$, $\quad x = u+v$, $y = uv$.

Let's find $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$.

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$= (ye^{xy})(1) + (xe^{xy})(v)$$

$$= uve^{(u+v)uv} + (u+v)ve^{(u+v)uv}$$

$$= (2uv+v^2)e^{(u+v)uv}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

$$= (ye^{xy})(1) + (xe^{xy})(u)$$

$$= uve^{(u+v)uv} + (u+v)ue^{(u+v)uv}$$

$$= (u^2+2uv)e^{(u+v)uv}$$