MAT 127 MIDTERM II

MONDAY APRIL 8, 2024
8:30–9:50PM

Name: ______________________________ ID: ______________

Lecture (circle yours):

L01	L02	L03
MW 4:00–5:20pm	TuTh 10:00–11:20am	TuTh 5:30–6:50pm

Instructions.

(1) Fill in your name and Stony Brook ID number and circle your lecture number at the top of this cover sheet.

(2) This exam is closed-book and closed-notes; no electronic devices.

(3) You have 80 minutes to complete this exam.

(4) Leave all answers in exact form (that is, *do not* approximate π, square roots, etc.)

(5) You must justify all your answers and show all your work. Even a correct answer without any justification will result in no credit.

Reference page.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots \quad |x| < 1$$

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad |x| < 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \quad x \in \mathbb{R}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad x \in \mathbb{R}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad x \in \mathbb{R}$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad |x| < 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = \binom{k}{0} + \binom{k}{1} x + \binom{k}{2} x^2 \binom{k}{3} x^3 + \cdots, \quad |x| < 1$$

$$\binom{k}{0} = 1, \quad \binom{k}{n} = \frac{k(k-1)\cdot \cdots \cdot (k-n+1)}{n!}$$

The **Taylor series of** $f(x)$ **centered at** $x = a$ is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

1. (a) (10 pts) Calculate the degree 3 Taylor polynomial $T_3(x)$ of $f(x) = e^{-x}$ centered around $x = 1$.

Solution. By the definition we have

$$T_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3.$$

We the compute the derivatives at 0:

$$\begin{aligned} f(x) &= e^{-x}, \quad f(1) = e^{-1} \\ f'(x) &= -e^{-x}, \quad f'(1) = -e^{-1} \\ f''(x) &= e^{-x}, \quad f''(1) = e^{-1} \\ f'''(x) &= -e^{-x}, \quad f'''(1) = -e^{-1}, \end{aligned}$$

which gives $T_3(x) = e^{-1} - e^{-1}(x-1) + \frac{e^{-1}}{2!}(x-1)^2 - \frac{e^{-1}}{3!}(x-1)^3$. □

(b) (10 pts) Find a power series representation of the integral

$$\int \frac{\sin(2x)}{x}\, dx.$$

Solution. Using the Maclaurin series for $\sin x$ we get

$$\sin(2x) = \sum_{n=0}^{\infty}(-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n \frac{2^{2n+1}x^{2n+1}}{(2n+1)!} = 2\sum_{n=0}^{\infty}(-1)^n \frac{4^n x^{2n+1}}{(2n+1)!}.$$

After dividing by x we get

$$\frac{\sin(2x)}{x} = 2\sum_{n=0}^{\infty}(-1)^n \frac{1}{x}\frac{4^n x^{2n+1}}{(2n+1)!} = 2\sum_{n=0}^{\infty}(-1)^n \frac{4^n x^{2n}}{(2n+1)!}.$$

We now compute the integral:

$$\begin{aligned} \int \frac{\sin(2x)}{x}\, dx &= \int 2\sum_{n=0}^{\infty}(-1)^n \frac{4^n x^{2n}}{(2n+1)!}\, dx = 2\sum_{n=0}^{\infty}(-1)^n \frac{4^n}{(2n+1)!}\int x^{2n}\, dx \\ &= C + 2\sum_{n=0}^{\infty}(-1)^n \frac{4^n}{(2n+1)!}\frac{x^{2n+1}}{2n+1}. \end{aligned}$$

□

2. (a) (10 pts) Verify that $y = Ce^{-x} + x - 1$ for any value of the constant C is a solution to the third order ODE $y''' + y'' = 0$.

Solution. We differentiate and find

$$y' = -Ce^{-x} + 1, \quad y'' = Ce^{-x}, \quad y''' = -Ce^{-x}.$$

Then $y''' + y'' = -Ce^{-x} + Ce^{-x} = 0$. □

(b) (10 pts) Find a solution to the initial-value problem

$$y''' + y'' = 0, \quad y(0) = 0.$$

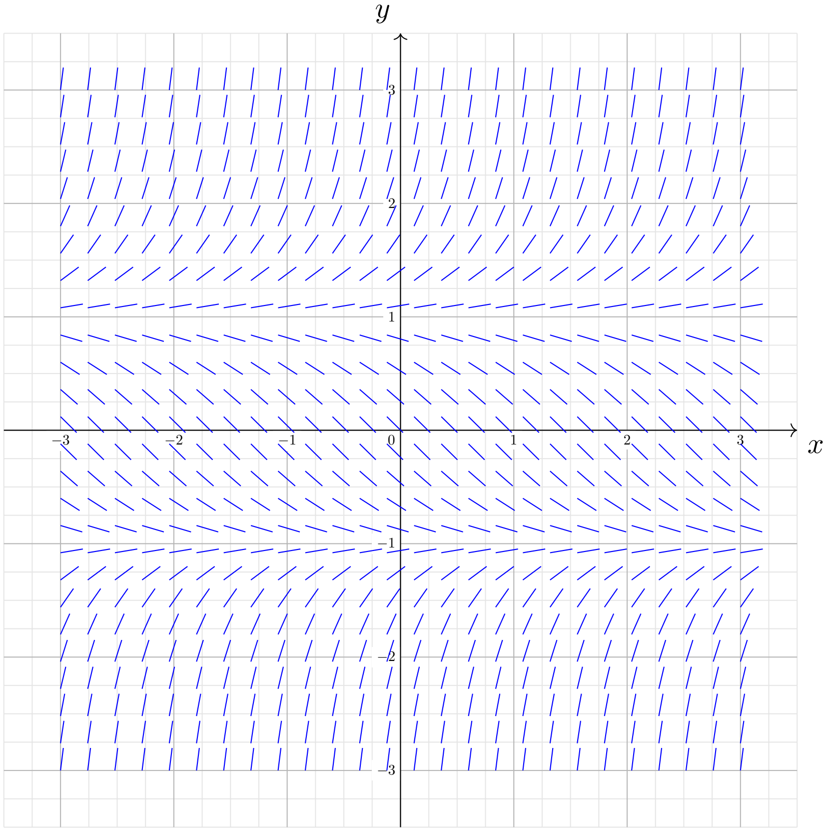
(You may use the result from part (a), even if you did not solve part (a).)

Solution. From part (a) we know that $y = Ce^{-x} + x - 1$ is a solution to the ODE $y''' + y'' = 0$. The specific solution that is sought satisfies $y(0) = 0$, which means

$$y(0) = Ce^0 + 0 - 1 = C - 1 = 0 \Leftrightarrow C = 1,$$

so a specific solution to the initial-value problem is $y = e^{-x} + x - 1$. □

3. (a) (10 pts) Sketch the slope field for the first order ODE $y' = y^2 - 1$.



(b) (10 pts) Use Euler's method with step size 1 to estimate $y(3)$ where $y(x)$ is the solution to the initial-value problem

$$y' = x^2 - y, \quad y(0) = 0.$$

Solution. We have $y(0) = 0$, so $(x_0, y_0) = (0, 0)$. The function for the slope is given by $F(x, y) = x^2 - y$. The first step is $x_1 = 1$, and

$$y_1 = y_0 + hF(x_0, y_0) = 0 + 1 \cdot (0^2 - 0) = 0.$$

Then we have $x_2 = 2$ and

$$y_2 = y_1 + hF(x_1, y_1) = 0 + 1 \cdot (1^2 - 0) = 1.$$

For the last step we have $x_3 = 3$ and

$$y_3 = y_2 + hF(x_2, y_2) = 1 + 1 \cdot (2^2 - 1) = 4,$$

so $y(3) \approx 4$. □

4. Find the general solution to the following separable first order ODEs
(a) (10 pts) $\frac{1}{3}yy' = x^2$

Solution. Write $y' = \frac{dy}{dx}$. Then we get

$$\frac{1}{3}y\frac{dy}{dx} = x^2 \Leftrightarrow \frac{1}{3}y\,dy = x^2\,dx \Leftrightarrow \int \frac{1}{3}y\,dy = \int x^2\,dx \Leftrightarrow \frac{1}{6}y^2 = \frac{1}{3}x^3 + C$$
$$\Leftrightarrow y^2 = 2x^3 + D \Leftrightarrow y = \pm\sqrt{2x^3 + D}.$$

□

(b) (10 pts) $\frac{dy}{dx} = ye^x$

Solution. We separate the variables and integrate.

$$\frac{dy}{dx} = ye^x \Leftrightarrow \frac{1}{y}\,dy = e^x\,dx \Leftrightarrow \int \frac{1}{y}\,dy = \int e^x\,dx \Leftrightarrow \log|y| = e^x + C$$
$$\Leftrightarrow y = e^{e^x + C} = e^C e^{e^x} = De^{e^x}.$$

□

5. Newton's law of cooling states that the rate of change of the temperature of is proportional to the difference between its own temperature and the ambient temperature.

When a pizza is removed from an oven it has the temperature 200 F and its change in temperature is described by Newton's law of cooling as follows:

$$\frac{dT}{dt} = -\frac{1}{10}(T - T_{\text{ambient}}).$$

The ambient temperature in the kitchen is 70 F. We would like to wait until the temperature of the pizza reaches 115 F before eating it so that we do not burn our tongues.

(a) (10 pts) Solve the initial-value problem

$$\frac{dT}{dt} = -\frac{1}{10}(T - 70), \quad T(0) = 200.$$

Solution. The ODE is separable, so we first find a general solution of it.

$$\begin{aligned}\frac{dT}{dt} = -\frac{1}{10}(T-70) \Leftrightarrow \frac{1}{T-70}\,dT = -\frac{1}{10}\,dt \Leftrightarrow \int \frac{1}{T-70}\,dT = \int -\frac{1}{10}\,dt \\ \Leftrightarrow \log|T-70| = -\frac{t}{10} + C \Leftrightarrow T - 70 = e^{-\frac{t}{10}+C} = De^{-\frac{t}{10}} \\ \Leftrightarrow T(t) = 70 - De^{-\frac{t}{10}}.\end{aligned}$$

Since we are given $T(0) = 200$ we can find the constant D. Namely $T(0) = 70 - De^{-\frac{0}{10}} = 70 - D = 200 \Leftrightarrow D = -130$. Therefore our solution is $T(t) = 70 + 130e^{-\frac{t}{10}}$. □

(b) (10 pts) Solve the equation $T(t) = 115$. (Leave the answer in its exact form.) This is how many minutes we have to wait until we eat the pizza.

Solution. The equation is equivalent to $70 + 130e^{-\frac{t}{10}} = 115 \Leftrightarrow 130e^{-\frac{t}{10}} = 45$. Now we divide both sides and take the natural logarithm to get

$$e^{-\frac{t}{10}} = \frac{45}{130} = \frac{9}{26} \Leftrightarrow -\frac{t}{10} = \log\left(\frac{9}{26}\right) \Leftrightarrow t = -10\log\left(\frac{9}{26}\right) = 10\log\left(\frac{26}{9}\right)$$

Therefore we have to wait $10\log\left(\frac{26}{9}\right)$ minutes (this number is approximately equal to 10.6). □